Math 400 - midterm

Closed book exam; no calculators. Adequately explain the steps you take and answer as much as you can (partial credit awarded).

1. Bugs on a circular petri dish have a density $u(r,\theta,t)$ that evolves according to

$$u_t - \gamma u = \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta}, \quad u(1,\theta,t)=0, \quad u(r,\theta,0)=f(r)\sin\theta.$$

where γ is a constant representing the bug birth rate and $f(r)$ denotes some initial distribution. Solve this problem using separation of variables, and provide a condition on γ that ensures that the bugs will survive for $t \gg 1$.

2. A biologist now takes the bugs and spreads them within a sphere, so that

$$u_t - \gamma u = \frac{1}{r^2}(r^2 u_r)_r, \quad u(1,t)=0, \quad u(r,0)=1, \quad u \text{ regular for } r \to 0.$$

Solve this problem using separation of variables. Is it now easier or harder for the bugs to survive?

Helpful information:

Fourier Series:

For a periodic function $f(x)$ with period $2L$, the Fourier series is

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right],$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx.$$

For any non-negative constant ν, **Bessel's equation**

$$z^2y'' + zy' + (z^2 - \nu^2)y = 0.$$

has a regular solution, $y(z) = J_\nu(z)$, with $J_\nu(z) \propto z^\nu$ for $z \to 0$, and a singular solution, $y(z) = Y_\nu(z)$. For $\nu = 1$, $J_1(z)$ has an infinite number of zeros, $z = z_n$, with the first at $z = z_1 \approx 3.83$. The more general ODE,

$$x^2y'' + (1-2\alpha)xy' + (\omega^2\beta^2x^{2\beta} + \alpha^2 - \nu^2\beta^2)y = 0,$$

with parameters α, ω, β and $\nu > 0$, has the solutions $x^\alpha J_\nu(\omega x^\beta)$ and $x^\alpha Y_\nu(\omega x^\beta)$. Note the special case

$$J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\sin z.$$

The **Sturm-Liouville ODE** is

$$[p(x)y']' + \lambda\sigma(x)y + q(x)y = 0, \qquad a < x < b,$$

with $\sigma(x) > 0$ and $p(x) > 0$. The associated expansion formula using the eigensolutions $\{\lambda_n, y_n(x)\}$ is

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x), \qquad c_n = \frac{\int_a^b f(x)y_n(x)\sigma(x)dx}{\int_a^b [y_n(x)]^2\sigma(x)dx}.$$

Helpful trig identities:

$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B \qquad \& \qquad \sin(A \pm B) = \sin A\cos B \pm \cos A\sin B.$$

Midterm exam - solution

1. (10 points) We separate variables for the PDE, $u = R(r)\Theta(\theta)T(t)$, finding

$$T_t = (\gamma - k^2)T, \qquad \Theta_{\theta\theta} = -m^2\Theta, \qquad r^2 X_{rr} + rX_r + \left(k^2r^2 - m^2\right) X = 0.$$

Given $u(r,\theta,t)$ must be $2\pi-$periodic in angle and regular for $r \to 0$, the solutions are $T \propto e^{(\gamma-k^2)t}$, $\Theta \propto \sin m\theta$ or $\cos m\theta$ if $m = 1, 2, ...$ or a constant if $m = 0$, and the Bessel function $J_m(kr)$ *(3 points)*. The boundary condition $R(1) = 0$ then implies that k must be a zero of $J_m(z)$. However, the initial condition is proportional to $\sin\theta$, so we may take $u(r,\theta,t) \propto \sin\theta$ and set $m = 1$. Moreover, $k = z_n$, the n^{th} (positive) zero of $J_1(z)$ (*i.e.* $J_1(z_n) = 0$) *(2 points)*. Hence, we write a solution suitable for the initial-value problem,

$$u = \sin\theta \sum_{n=1}^{\infty} c_n J_1(z_n r) e^{(\gamma - z_n^2)t}, \qquad c_n = \int_0^1 f(r) J_1(z_n r) r dr \;\times\; \left[\int_0^1 [J_1(z_n r)]^2 r dr\right]^{-1}$$

using the Sturm-Liouville expansion formula, and given that the radial problem is of SL form with weight function $\sigma = r$ and boundary conditions of type 2 (regularity) at $r = 0$ and type 1 (mixed, or Dirichlet) at $r = 1$ *(3 points)*. For large times, the first term of the series dominates with time-dependence $e^{(\gamma - z_1^2)t}$. Thus, bugs die out unless $\gamma > z_1^2 \approx 3.83^2$ *(2 points)*.

2. (10 points) We separate variables: $u(r,t) = R(r)T(t)$, giving

$$\frac{T'}{T} - \gamma = \frac{(r^2R')'}{r^2R} = -\lambda.$$

Hence $T(t)$ is proportional to $\exp(\gamma - \lambda)t$ again *(1 point)*. The ODE for $R(r)$ can be written as

$$r^2R'' + 2rR' + \omega^2r^2R = 0,$$

with $\lambda = \omega^2$, which is a Sturm-Liouville ODE with weight function $\sigma(r) = r^2$ and boundary conditions of type 2 (regularity) at $r = 0$ and type 1 (mixed, or Dirichlet) at $r = 1$ *(2 points)*. It is also Bessel's equation in disguise – we have the more general ODE quoted in the helpful information, with

$$\alpha = -\frac{1}{2}, \quad \beta = 1, \quad \nu = \frac{1}{2}, \quad R(r) \propto r^{-\frac{1}{2}} J_{\frac{1}{2}}(\omega r) \propto \frac{\sin(\omega r)}{r}$$

(which does not diverge for $r \to 0$). But the boundary condition $R(1) = 0$ implies that $J_\nu(\omega) = 0$. *i.e.* $\omega = n\pi$. *(5 points)*

Bearing in mind the initial conditions, we now write the solution

$$u(r,t) = \frac{1}{r}\sum_{n=0}^{\infty} c_n \sin(n\pi r) e^{(\gamma - n^2\pi^2)t}, \quad c_n = \frac{\int_0^1 \sin(n\pi r) r \; dr}{\int_0^1 \sin^2(n\pi r) \; dr} = \frac{2[1-(-1)^n]}{n\pi},$$

in view of the Sturm-Liouville expansion formula *(2 points)*. It is easier to survive as $\pi < 3.83$ *(1 point)*.