Coursework 5: Method of characteristics

(1) Using the method of characteristics, solve

$$u_t - 3xt^2u_x = t^2x^2,$$

for $-\infty < x < \infty$, subject to $u(x,0) = \cos x$.

(2) Using the method of characteristics, solve

$$u_t + (t - x)u_x = x,$$

for $x \geq 0$, subject to $u(x,0) = F(x)$ and $u(0,t) = f(t)$.

(3) Solve

$$u_t + (1+u)u_x = 0, \quad -\infty < x < \infty, \quad t > 0, \quad u(x,0) = \sigma \tan^{-1} x,$$

for (a) $\sigma = +1$ and (b) $\sigma = -1$. In each case, establish whether the solution continues to exist for all time, finding when and where things go wrong if all goes pear-shaped. Sketch the characteristics diagram and sample snapshots of your solution, indicating any breakdown of the solution.

Bonus: For the case with a shock, sketch a solution for $t = 20$ in which an equal-area rule is used to avoid a multi-valued solution. Find the shock speed, and suggest what form the solution takes in the long-time limit.

Sample problems

(1) Using the method of characteristics, solve

$$x^2 u_t + u_x = u^{-2}$$

for $-\infty < x < \infty$ and $t > 0$, subject to $u(x,0) = f(x)$.

The characteristic equations are

$$\frac{dx}{dt} = \frac{1}{x^2} \quad \& \quad \frac{du}{dt} = -\frac{1}{x^2u^2}.$$

Hence, given that $x = x_0$ *and* $u = f(x_0)$ *at* $t = 0$,

$$x^3 = x_0^3 + 3t \quad \& \quad u^3 = [f(x_0)]^3 + 3(x_0^2 + 3t)^{1/3} - 3x_0.$$

Replacing x_0 *by* $(x^3 + 3t)^{1/3}$ *in the latter gives* $u(x,t)$.

(2) Using the method of characteristics, solve

$$u_t + x(1-x)u_x = x,$$

for $-\infty < x < \infty$, subject to $u(x,0) = 0$.

The characteristic equations are

$$\frac{dx}{dt} = x(1-x) \quad \& \quad \frac{du}{dt} = x.$$

Hence, given that $x = x_0$ *and* $u = 0$ *at* $t = 0$,

$$x = \frac{x_0 e^t}{1 - x_0 + x_0 e^t} \quad \text{or} \quad x_0 = \frac{x}{x + (1-x)e^t} \quad \& \quad u = \log[1 + x_0(e^t - 1)] = \log\left[1 + \frac{x(e^t - 1)}{x + (1-x)e^t}\right].$$

(3) Using the method of characteristics, solve

$$x^2 u_t + u_x = x^2$$

for $x > 0$, subject to $u(x,0) = 0$ and $u(0,t) = f(t)$. Provide a condition on $f(t)$ which guarantees that the solution is continuous.

The characteristic equations are

$$\frac{dx}{dt} = x^{-2} \quad \& \quad \frac{du}{dt} = 1.$$

Hence, if $x = x_0$ *and* $u = 0$ *at* $t = 0$,

$$x^3 = x_0^3 + 3t \quad \& \quad u = t$$

which is the case for $x^3 > 3t$. *But if the characteristic leaves* $x = 0$ *at* $t = t_0$ *with* $u = f(t_0)$, *we find instead that*

$$x^3 = 3(t - t_0) \quad \& \quad u = f(t_0) + t - t_0 = f(t - x^3/3) + x^3/3.$$

If $f(0) = 0$, *then* $u \to x^3/3 = t$ *along the dividing characteristic curve* $t = x^3/3$, *rendering the solution continuous.*

Coursework 5: Solutions to actual problems

(1) (2 points) The characteristic equations are

$$\frac{dx}{dt} = -3xt^2 \quad \& \quad \frac{du}{dt} = t^2x^2,$$

giving

$$x = x_0 e^{-t^3} \quad \& \quad u = \cos x_0 + \frac{1}{6}x_0^2(1 - e^{-2t^3}).$$

Eliminating x_0 gives

$$u = \cos(xe^{t^3}) + \frac{1}{6}x^2(e^{2t^3} - 1).$$

(2) (5 points) The characteristic equations are

$$\frac{dx}{dt} = t - x \quad \& \quad \frac{du}{dt} = x.$$

Hence, if the characteristic intersects $x = x_0$ and $u = F(x_0)$ at $t = 0$,

$$x = (x_0 + 1)e^{-t} + t - 1 \quad \& \quad u = F(x_0) + x_0 + 1 + \frac{t^2}{2} - t - (x_0 + 1)e^{-t},$$

giving

$$u(x, t) = F((x - t + 1)e^t - 1) + (x - t + 1)e^t + \frac{t^2}{2} - (x + 1) \qquad \text{for } x > e^{-t} + t - 1.$$

But if the characteristic leaves $x = 0$ at $t = t_0$ with $u = f(t_0)$, we find instead, for $x < e^{-t}+t-1$, that

$$x = (1 - t_0)e^{t_0 - t} + t - 1 \quad \& \quad u = f(t_0) + \frac{1}{2}(t^2 - t_0^2) - t - (1 - t_0)e^{t_0 - t} + 1,$$

for which the solution is given only implicitly.

(3) (8 points) The characteristic equations are

$$\frac{dx}{dt} = 1 + u \quad \& \quad \frac{du}{dt} = 0.$$

Hence, given $u = f(x_0)$ at $t = 0$,

$$x = x_0 + (1 + u)t \quad \text{and} \quad u = \sigma \tan^{-1}[x - (1 + u)t].$$

That is, the implicit solution, $x = \sigma \tan u + (1 + u)t$, which can be graphed easily at least.

For $\sigma = +1$, the solutions becomes shallower with time, translating to the right. With $\sigma = -1$, however, the steepen whilst translating. A shock forms when u_x first diverges:

$$(\sigma \sec^2 u + t)u_x = 1$$

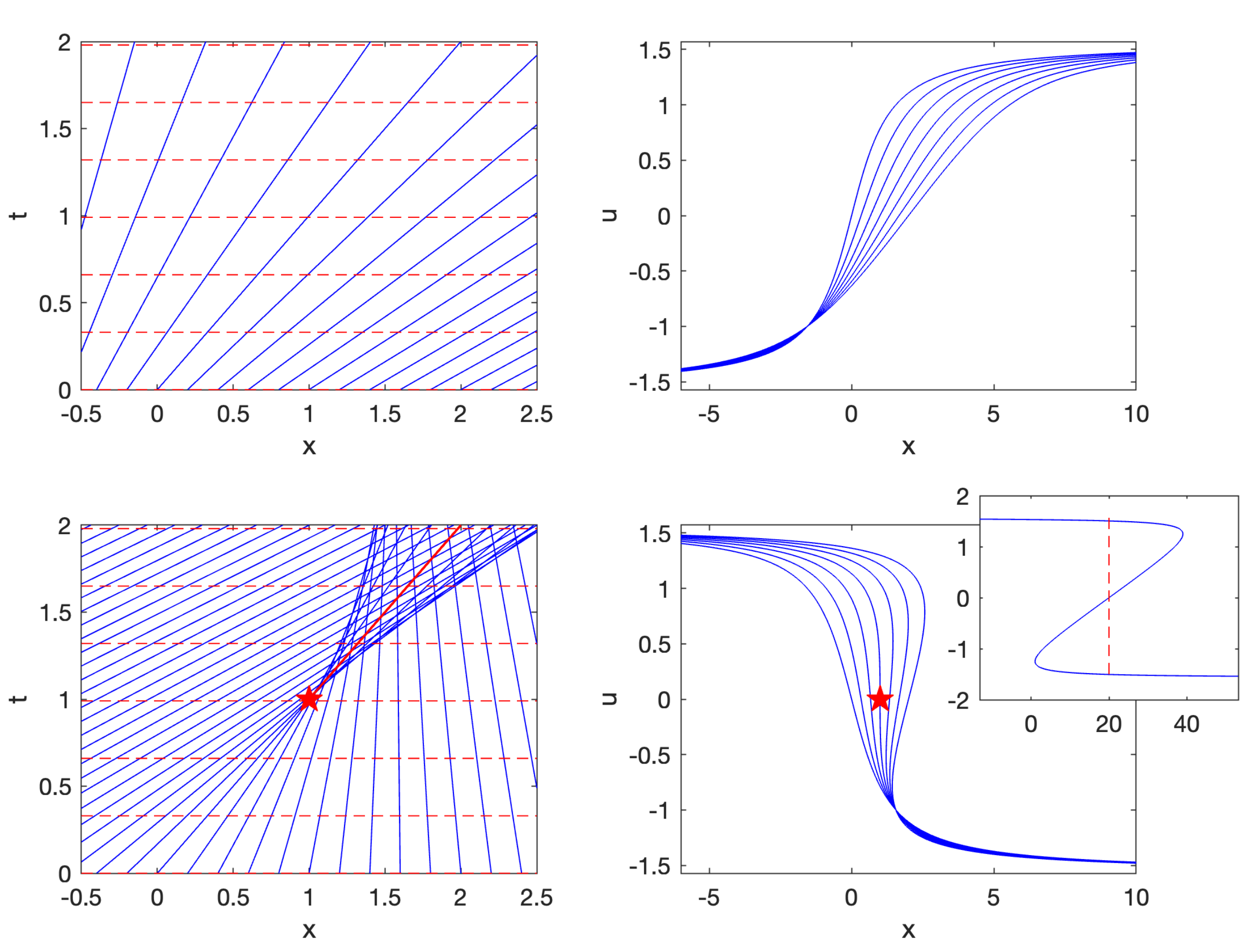
The minimum t for which u_x first diverges is therefore at the maximum of $\sec^2 u$ when $\sigma = -1$. *i.e.* a shock forms at $x = t = 1$. Afterwards, the solution is multi-valued.

For the bonus (4 points): The shock speed is

$$\frac{dX}{dt} = -\frac{u^+(u^+ - 2) - u^-(u^- - 2)}{2(u^+ - u^-)} = 1,$$

since the profile is symmetrical about $u = 0$, implying $u^+ = -u^-$. For large times, $u_\pm$ must approach $\mp\frac{1}{2}\pi$, so the solution approaches the step function,

$$u = \frac{1}{2}\pi \operatorname{sgn}(t - x).$$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