Coursework 4: Transforms

(1) Using Fourier transforms, find $\hat{f}(k) = \mathcal{F}\{u(x,0)\}$ and solve the heat equation on the infinite line ($-\infty < x < \infty$) subject to the initial conditions

$$(a) \qquad u(x,0) = e^{-3(x+a)^2},$$

$$(b) \qquad u(x,0) = x[\delta(x-a) + \delta(x+a)],$$

$$(c) \qquad u(x,0) = \sin(x+a),$$

where $\delta(x)$ is Dirac's delta function and a is a positive parameter.

(2) Consider the integral equation for $f(x)$,

$$\frac{1}{a^2 + x^2} = \int_{-\infty}^{\infty} \frac{f(x-u)du}{b^2 + u^2},$$

where a and b are positive parameters. By using the Fourier transform show that there is only a solution with the form of a regular function if $a > b$. Find that solution. What is the solution if $a = b$?

(3) The Fourier sine transform and its inverse are

$$\mathcal{F}_S\{f(x)\} = \int_0^{\infty} f(x)\sin(kx)dx, \qquad \mathcal{F}_S^{-1}\{f(x)\} = \frac{2}{\pi}\int_0^{\infty} f(x)\sin(kx)dk.$$

If $u \to 0$ and $u_x \to 0$ for $x \to \infty$, show that

$$\mathcal{F}_S\{u_{xx}(x,t)\} = -k^2\mathcal{F}_S\{u(x,t)\} + ku(0,t).$$

and

$$\mathcal{F}_S\{e^{-ax}\} = \frac{k}{a^2 + k^2} \quad (a > 0).$$

Use the Fourier sine transform to write the solution to

$$u_t = u_{xx} + e^{-ax-t}, \qquad 0 \le x < \infty, \qquad u(0,t) = u(x,0) = 0,$$

in terms of an inverse sine transform.

(4) Establish that $\mathcal{L}\{e^{at}\} = (s-a)^{-1}$, $\mathcal{L}\{t^n\} = s^{-n-1}n!$ and $\mathcal{L}\{f(t-a)H(t-a)\} = e^{-as}\overline{f}(s)$. Use a Laplace transform to solve

$$u_x + xu_t = xt^n,$$

for $x \ge 0$, subject to $u(x,0) = 0$ and $u(0,t) = t^n e^{-t}$.

(5) Establish that $\mathcal{L}\{e^{-t}\delta(x-t)\} = e^{-(s+1)x}$, where $\delta(x)$ is Dirac's delta-function. Use a Laplace transform to solve

$$u_{tt} = u_{xx} + e^{-t}\delta(x-t), \qquad 0 \le x < \infty, \qquad u(0,t) = u(x,0) = u_t(x,0) = 0 \ \& \ u \to 0 \text{ as } x \to \infty.$$

Fourier transforms; warm-ups

(1) Solve the heat equation on the infinite line ($-\infty < x < \infty$) subject to the initial conditions

$$(a) \quad u(x,0) = e^{-x^2/4}, \qquad (b) \quad u(x,0) = -\frac{x}{2}e^{-x^2/4},$$

$$(c) \quad u(x,0) = \delta(x), \qquad (d) \quad u(x,0) = \sin \kappa x,$$

where $\delta(x)$ is Dirac's delta function and κ is a constant.

Fourier transforming the heat equation and integrating implies that $\hat{u}(k,t) = \hat{f}(k)e^{-k^2 t}$, *or*

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} f(y) e^{-(x-y)^2/4t} dy.$$

For (a), the latter gives

$$u(x,t) = \frac{e^{-x^2/4(1+t)}}{\sqrt{1+t}}.$$

The solution to (b) is the $x-$*derivative of the solution to (a)! For (c), using the properties of the delta-function, we find*

$$u(x,t) = \frac{e^{-x^2/4t}}{\sqrt{4\pi t}}.$$

For (d), we have $\hat{f}(k) = -i\pi[\delta(k-\kappa) - \delta(k+\kappa)]$, *and so* $u = e^{-\kappa^2 t} \sin \kappa x$.

(2) Solve the heat equation on the infinite line ($-\infty < x < \infty$) subject to the initial conditions

$$u(x,0) = e^{-x^2/2} \sin \gamma x$$

for some parameter γ.

We know, or can compute

$$\mathcal{F}\{e^{-x^2}\} = \sqrt{\pi} e^{-k^2/4}$$

The shifting theorems, $\mathcal{F}\{f(ax)\} = \hat{f}(k/a)/|a|$ *and* $\mathcal{F}\{e^{iax}f(x)\} = \hat{f}(k-a)$ *therefore imply that*

$$\mathcal{F}\{e^{-x^2/2}\} = \sqrt{2\pi} e^{-k^2/2} \qquad \mathcal{F}\{e^{\pm i\gamma x - x^2/2}\} = \sqrt{2\pi} e^{-(k\pm\gamma)^2/2}.$$

Hence

$$\mathcal{F}\{e^{-x^2/2} \sin \gamma x\} = -i\sqrt{2\pi} e^{-(k^2+\gamma^2)/2} \sinh \gamma k.$$

Thus,

$$u(x,t) = \frac{1}{i\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx - (k^2+\gamma^2)/2 - k^2 t} \sinh(\gamma k)\, dk$$

Finally, we use the fact that

$$\int_{-\infty}^{\infty} e^{-ak^2 + ikX + kY} dk = e^{(iX+Y)^2/4a} \int_{-\infty - iX/2a}^{\infty - iX/2a} e^{-az^2} dz = e^{(iX+Y)^2/4a} \sqrt{\frac{\pi}{a}},$$

equivalent to what was established in class, to obtain

$$u(x,t) = \frac{1}{\sqrt{1+2t}} \exp\left[\frac{(\gamma^2 - x^2)}{2(1+2t)} - \frac{\gamma^2}{2}\right] \sin\left(\frac{\gamma x}{1+2t}\right).$$

(3) Solve

$$u_t = -tu_x, \qquad -\infty < x < \infty, \qquad u(x,0) = f(x).$$

Verify your solution by direct substitution into the PDE.

Fourier transforming:

$$\hat{u}_t = -ikt\hat{u}, \qquad \rightarrow \qquad \hat{u}(k,t) = \hat{f}(k)e^{-ikt^2/2}$$

in view of the transformed initial condition. Hence

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)e^{ikx-ikt^2/2}dk$$

$$\equiv f(x - t^2/2)$$

We have $u_t = -tf'(x-t^2/2)$ *and* $u_x = f'(x-t^2/2)$ *by the chain rule; hence* $u_t = -tu_x$.

(4) For prescribed $g(x)$ and $K(x)$,

$$g(x) = \int_{-\infty}^{\infty} K(x-y)f(y)dy$$

defines an integral equation for $f(x)$. Solve this equation by first taking the Fourier transform, and finding an expression for $\hat{f}(k)$, and then undoing the Fourier transform. If $K(x) = ag(x-b)$, for some constants a and b, what is $f(x)$? Find the solution if $K(x) = g - g_{xx}$.

The Fourier transform indicates that

$$\hat{g}(k) = \hat{K}(k)\hat{f}(k).$$

Hence

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\hat{g}(k)}{\hat{K}(k)}e^{ikx}dk.$$

If $K(x) = ag(x-b)$, *then* $\hat{K}(k) = ae^{-ikb}\hat{g}(k)$ *(using a shifting theorem) and so* $f(x) = \delta(x+b)/a$, *using the definition of Dirac's delta function.*

If $K(x) = g - g_{xx}$, *then* $\hat{K}(k) = (1+k^2)\hat{g}(k)$, *and so* $f = \mathcal{F}^{-1}\{(1+k^2)^{-1}\}$. *This inverse transform can be determined by either noting that* $\mathcal{F}\{e^{-|x|}\} = 2/(1+k^2)$, *or by direct computation of the integral (achieved by extending it to an infinite semi-circular arc on the complex plane and evaluating the residues of the poles at* $k = \pm i$, *depending on which is enclosed). Thence,* $f(x) = e^{-|x|}/2$.

(5) The Fourier cosine transform and its inverse are

$$\mathcal{F}_C\{f(x)\} = \int_0^{\infty} f(x)\cos(kx)dx, \qquad \mathcal{F}_C^{-1}\{f(x)\} = \frac{2}{\pi}\int_0^{\infty} f(x)\cos(kx)dk.$$

If $u \to 0$ and $u_x \to 0$ for $x \to \infty$, show that

$$\mathcal{F}_C\{u_{xx}(x,t)\} = -k^2\mathcal{F}_C\{u(x,t)\} - u_x(0,t)$$

and

$$\mathcal{F}_C\{e^{-ax}\} = \frac{a}{a^2+k^2}.$$

Use the Fourier cosine transform to write the solution to

$$u_t = u_{xx}, \qquad 0 \le x < \infty, \qquad u_x(0,t) = 0, \qquad u(x,0) = e^{-x},$$

in terms of an inverse cosine transform.

Using the definitions and integration by parts,

$$\mathcal{F}_C\{u_{xx}\} = \int_0^\infty u_{xx}\cos(kx)dx = -u_x(0,t) + k\int_0^\infty u_x \sin(kx)dx$$

$$= -u_x(0,t) - k^2\int_0^\infty u\cos(kx)dx = -u_x(0,t) - k^2\mathcal{F}_C\{u\}.$$

Then,

$$\mathcal{F}_C\{e^{-ax}\} = \int_0^\infty e^{-ax}\cos(kx)dx = \frac{1}{a} - \frac{k}{a}\int_0^\infty e^{-ax}\sin(kx)dx = \frac{1}{a} - \frac{k^2}{a^2}\int_0^\infty e^{-ax}\cos(kx)dx,$$

giving the needed result.

Applying the cosine transform, we find

$$\frac{\partial}{\partial t}\hat{u}_C = -k^2\hat{u}_C \qquad \rightarrow \qquad \hat{u}_C(k,t) = \frac{e^{-k^2 t}}{1+k^2},$$

where $\hat{u}_C(k,t) = \mathcal{F}_C\{u(x,t)\}$*. Hence*

$$u(x,t) = \mathcal{F}_C^{-1}\left\{\frac{e^{-k^2 t}}{1+k^2}\right\}.$$

Laplace transforms - warm-ups

(1) Establish that $\mathcal{L}\{t^n\} = s^{-n-1}n!$ and $\mathcal{L}\{f(t-a)H(t-a)\} = e^{-as}\overline{f}(s)$. Use a Laplace transform to solve

$$x^2 u_t + u_x = x^2,$$

for $x \geq 0$, subject to $u(x,0) = 0$ and $u(0,t) = f(t)$.

Solution: Inserting t^n into the definition of the Laplace transform and integrating gives the first result (as long as $\mathrm{Re}(s) > 0$). Inserting the second function into the definition and then changing the integration variables gives the second. Laplace transforming the PDE and boundary condition:

$$sx^2\overline{u} + \overline{u}_x = \frac{x^2}{s}, \quad \overline{u}(0,s) = \overline{f}(s).$$

Hence

$$\overline{u} = [\overline{f}(s) - s^{-2}]e^{-sx^3/3} + s^{-2}.$$

Inverting the transform with the help of the shifting theorem:

$$u(x,t) = t + \left[x^3/3 - t + f(t - x^3/3)\right] H\left(t - x^3/3\right).$$

(2) Compute $\mathcal{L}\{e^{-|t-a|}\}$ for $a > 0$. Use a Laplace transform to solve

$$u_t + cu_x = ce^{-|x-t|} \qquad u(0,t) = u(x,0) = 0 \;\&\; u \to 0 \text{ as } x \to \infty,$$

for $c \neq 1$ and $c = 1$.

Solution: We have

$$\mathcal{L}\{e^{-|t-a|}\} = \int_0^a e^{-st-a+t}dt + \int_a^\infty e^{-st+a-t}dt = \frac{e^{-a} - e^{-sa}}{s-1} + \frac{e^{-sa}}{1+s}.$$

Laplace transforming the PDE:

$$\overline{u}_x + \frac{s}{c}\overline{u} = \frac{e^{-x} - e^{-sx}}{s-1} + \frac{e^{-sx}}{1+s}.$$

Hence if $c \neq 1$,

$$\overline{u}(x,s) = \frac{c(e^{-x} - e^{-sx/c})}{(1-c)}\left(\frac{1}{s-1} - \frac{1}{s-c}\right) + \frac{c(e^{-sx} - e^{-sx/c})}{s(1-c)}\left(\frac{1}{s+1} - \frac{1}{s-1}\right)$$

Inverting the transform, and using the shifting theorem gives

$$u(x,t) = \frac{c}{(1-c)}\left[e^{t-x} - e^{ct-x} + H(t-x)(2 - e^{x-t} - e^{t-x}) + H(t-x/c)(e^{ct-x} + e^{x/c-t} - 2)\right].$$

For $c = 1$, we have

$$\overline{u}(x,s) = \frac{e^{-x} - e^{-sx}}{(s-1)^2} + xe^{-sx}\left(\frac{1}{s+1} - \frac{1}{s-1}\right)$$

which gives

$$u(x,t) = te^{t-x} + (xe^{x-t} - te^{t-x})H(t-x)$$

(3) Establish that $\mathcal{L}\{e^{at}\} = (s-a)^{-1}$. Use a Laplace transform to solve

$$u_t + xu_x = x^2,$$

for $x \geq 0$, subject to $u(x, 0) = 0$ and $u(0, t) = 0$.

Solution: Inserting the function into the definition of the Laplace transform and integrating gives the desired result (as long as $\text{Re}(s) > a$). Laplace transforming the PDE and boundary condition:

$$s\overline{u} + x\overline{u}_x = \frac{x^2}{s}, \quad \overline{u}(0, s) = 0.$$

Hence $\overline{u} = x^2/[s(s+2)]$ (using an integrating factor of x^s, and then the boundary condition to discard the homogeneous solution). Inverting the transform using a partial fraction gives

$$u(x, t) = \frac{1}{2}x^2(1 - e^{-2t}).$$

(4) Establish that $\mathcal{L}\{\cos at\} = s/(s^2 + a^2)$ and $\mathcal{L}\{\sin at\} = a/(s^2 + a^2)$. Use a Laplace transform to show that the solution to

$$u_t + cu_x = \cos\omega t\, \delta(x - t), \qquad u(0, t) = u(x, 0) = 0 \;\&\; x > 0,$$

for $c > 1$ is

$$u(x, t) = \frac{\cos[\Omega(x - ct)/c][H(ct - x) - H(t - x)]}{c - 1}.$$

where $\Omega = \omega c/(c - 1)$. Show that $u(x, t) = \omega^{-1} \sin\omega t\, \delta(t - x)$ for $c = 1$.

Solution: Inserting the functions into the definition of the Laplace transform and integrating by parts connects the transforms together and then gives the desired result (as long as $\text{Re}(s) > 0$). Laplace transforming the PDE and boundary condition:

$$c\overline{u}_x + s\overline{u} = e^{-sx}\cos\omega x, \qquad \overline{u}(0, s) = 0.$$

Hence

$$\overline{u}(x, s) = \frac{s(e^{-sx/c} - e^{-sx}\cos\omega x) + \Omega e^{-sx}\sin\omega x}{(c - 1)(s^2 + \Omega^2)}.$$

Inverting the transform and using a trig relation gives the first result. For $c = 1$, we find $\overline{u}(x, s) = \omega^{-1}e^{-sx}\sin\omega x$, and inverting the transform gives the second result.

Actual Solutions

(1) (10 points) Fourier transforming the heat equation and integrating implies that $\hat{u}(k,t) = \hat{f}(k)e^{-k^2 t}$. Useful results are

$$\mathcal{F}\{e^{-ax^2}\} = \sqrt{\frac{\pi}{a}}e^{-k^2/4a}, \quad \mathcal{F}\{f(x-a)\} = e^{-ika}\hat{f}(k), \quad \int_{-\infty}^{\infty} \delta(x-a)F(x)dx = F(a),$$

$$\mathcal{F}\{e^{\pm ix}\} = 2\pi\delta(k \mp 1)$$

(4 points). Hence,

$$(a) \quad \hat{f}(k) = \sqrt{\frac{\pi}{3}}e^{ika-k^2/12)}, \qquad u(x,t) = \frac{1}{\sqrt{1+12t}}\exp\left[-\frac{3(x+a)^2}{1+12t}\right]$$

(2 points).

$$(b) \quad \hat{f}(k) = -2ia\sin ka, \qquad u(x,t) = \frac{a}{2\sqrt{\pi t}}\left[e^{-(x-a)^2/4t} - e^{-(x+a)^2/4t}\right]$$

(2 points).

$$(c) \quad \hat{f}(k) = -i\pi e^{ia}\delta(k-1) + i\pi e^{-ia}\delta(k+1), \quad u(x,t) = e^{-t}\sin(x+a)$$

(2 points).

(2) (4 points) The Fourier Transform of the integral equation is

$$\mathcal{F}\{\frac{1}{x^2+a^2}\} = \hat{f}(k)\mathcal{F}\{\frac{1}{x^2+b^2}\} \qquad \text{or} \qquad \hat{f}(k) = \frac{b}{a}e^{-(a-b)|k|}$$

which will only give a regular function for $f(x)$ on inverting the transform if $a > b$ *(2 points)*. Then,

$$f(x) = \frac{b(a-b)}{\pi a[(a-b)^2 + x^2]}.$$

For $a = b$, $\hat{f}(k) = 1$ and so $f(x) = \delta(x)$ *(2 points)*.

(3) (4 points) Using the definitions and integration by parts,

$$\mathcal{F}_S\{u_{xx}\} = \int_0^{\infty} u_{xx}\sin(kx)dx = -k\int_0^{\infty} u_x\cos(kx)dx = ku(0,t) - k^2\int_0^{\infty} u\sin(kx)dx,$$

leading to the first result. Then,

$$\mathcal{F}_S\{e^{-ax}\} = \int_0^{\infty} e^{-ax}\sin(kx)dx = \frac{k}{a}\int_0^{\infty} e^{-ax}\cos(kx)dx = \frac{k}{a^2} - \frac{k^2}{a^2}\int_0^{\infty} e^{-x}\sin(kx)dx,$$

giving the other needed result *(2 points)*. Applying the sine transform to the PDE, we find

$$\frac{\partial}{\partial t}\hat{u}_S + k^2\hat{u}_S = \frac{ke^{-t}}{a^2+k^2} \qquad \rightarrow \qquad \hat{u}_S(k,t) = \frac{k(e^{-t} - e^{-k^2 t})}{(a^2+k^2)(k^2-1)}$$

where $\hat{u}_S(k,t) = \mathcal{F}_S\{u(x,t)\}$, and then $u(x,t) = \mathcal{F}_S^{-1}\{\hat{u}_S(k,t)\}$ *(2 points)*.

(4) (5 points) Inserting the functions into the definition of the Laplace transform and integrating, integrating by parts, or changing variables, gives all the desired results (as long as $\mathrm{Re}(s) > a$ and $\mathrm{Re}(s) > 0$, for the first two, respectively) *(1 point)*. Laplace transforming the PDE and boundary condition:

$$\overline{u}_x + sx\overline{u} = \frac{xn!}{s^{n+1}}, \quad \overline{u}(0,s) = \frac{n!}{(s+1)^{n+1}}.$$

Hence, using the integrating factor $e^{sx^2/2}$,

$$\overline{u} = n! e^{-sx^2/2} \left[\frac{1}{(s+1)^{n+1}} - \frac{1}{s^{n+2}} \right] + \frac{n!}{s^{n+2}}$$

(3 points). Inverting the transform using the shifting theorem:

$$u(x,t) = \left[\left(t - x^2/2\right)^n e^{x^2/2 - t} - \frac{\left(t - x^2/2\right)^{n+1}}{n+1} \right] H(t - x^2/2) + \frac{t^{n+1}}{n+1}$$

(2 points).

(5) (5 points) Inserting the function into the definition of the Laplace transform gives the desired result *(1 point)*. Laplace transforming the PDE gives

$$\overline{u}_{xx} - s^2 \overline{u} = -e^{-(s+1)x}.$$

Hence,

$$\overline{u}(x,s) = \frac{e^{-sx}(1 - e^{-x})}{1 + 2s}$$

given that $\overline{u} \to 0$ for $x \to \infty$, which rules out the solution e^{sx} when $\mathrm{Re}(s) > 0$ *(3 points)*. Inverting the transform gives

$$u(x,t) = \frac{1}{2}(1 - e^{-x}) e^{(x - t/2)} H(t - x)$$

(1 point).