**MATH 400 – Final exam**

*For each question, provide 6 brief answers (i.e. one or two lines/sentences per answer); the detailed workings leading to each are not needed, but be sure to make certain that everything is defined and understandable in your answers. The questions are in black. Blue describes details of each problem. Red denotes bonuses for extra points.*

**1.** Chemical waste released into a river from a factory at $x = 0$ has concentration $u(x, t)$. The amount of the chemical released is $u(0, t) = C(t)$. In the river, the chemical is advected in $x > 0$ and depleted according to

$$u_t + v(x)u_x = -\mu u, \tag{1}$$

where $v(x)$ is the river flow speed and $\mu > 0$ is the (constant) depletion rate. The river was clean initially, $u(x, 0) = 0$, and the total amount of pollutant in it is

$$U(t) = \int_0^\infty u(x,t)dx.$$

*(i)* Provide a formula for $\overline{u}(x, s) = \mathcal{L}\{u(x, t)\}$ in terms of $\overline{C}(s) = \mathcal{L}\{C(t)\}$ and

$$T(x) = \int_0^x \frac{d\hat{x}}{v(\hat{x})},$$

and hence write down a solution for $u(x, t)$ in terms of $C(t)$ and $T(x)$.

By monitoring the river, the factory attempts to control the environmental impact of the chemical release by demanding that $C(t) = P(t) - U(t)$, for some base production function $P(t) > 0$.

*(ii)* Provide a formula for $\overline{C}(s)$.

Now take $v(x) = \frac{1}{2} + x$.

*(ii)* Given that $\mathcal{L}\{\int_0^t f(t-\tau)g(\tau)d\tau\} = \overline{f}(s)\overline{g}(s)$, write down the solution for $C(t)$ as a convolution integral.

For the last three parts, consider the case that $P(t) = \mathit{constant}$:

*(iv)* Provide explicit solutions for $\overline{C}(s)$, $C(t)$ and $u(x, t)$.

*(v)* Establish the limits of $C(t)$ and $u(x, t)$ for $t \to \infty$, and compare these limits with the steady solution $u = w(x)$ to (1).

*(vi)* Compare the limits found in *(v)* with the limits of $s\overline{C}(s)$ and $s\overline{u}(x, s)$ for $s \to 0$, rationalizing from the definition of the Laplace transform why they are related, or if not, why they are not.

**2.** Consider the PDE,

$$u_t + (u^2 - 1)u_x = -u, \qquad u(x, 0) = f(x).$$

*(i)* Write an implicit solution for $u(x, t)$.

*(ii)* Provide a condition on the initial condition $f(x)$ that the solution does not form a shock.

*(iii)* Determine the time and position at which that discontinuity first forms if $f(x) = Ae^{\frac{1}{4}-x^2}$. Locate this position for $A \gg 1$.

*(iv)* Provide a formula for the speed of any shock that forms in terms of the limits of $u$ from the left, $u^-$, and right, $u^+$.

*(v)* Establish the path of the shock in the case $f(x) = H(-x)$ (where $H(x)$ is Heaviside's step function).

*(vi)* Provide an explicit solution for $f(x) = H(x) - H(\frac{1}{3} - x)$. Without solving the problem in full, comment on how and when the solution becomes qualitatively different if $f(x) = H(x) - H(\frac{1}{6} - x)$.