Algebraic problems

1. Find the rescalings for the roots of

$$\epsilon^5 x^3 - (3 - 2\epsilon^2 + 10\epsilon^5 - \epsilon^6)x^2 + (30 - 3\epsilon - 20\epsilon^2 + 2\epsilon^3 + 24\epsilon^5 - 2\epsilon^6 - 2\epsilon^7)x - 72 + 6\epsilon + 54\epsilon^2 = 0,$$

and thence find two (non-trivial) terms in the approximation for each root, using (a) iteration and (b) expansion.

2. Develop two terms of the perturbation solutions to

$$\delta x^3 - (1 - 2\delta + \delta^3)x^2 + (1 - 3\delta - \delta^2 - 6\delta^3 + 2\delta^4)x + 2\delta - 3\delta^2 - 11\delta^3 + 6\delta^4 = 0,$$

for $\delta \ll 1$ and $\delta \gg 1$.

3. Develop perturbation solutions to

$$x^3 + (3 + 4\epsilon + \epsilon^2)x^2 + (3 + 9\epsilon + 7\epsilon^2 + 2\epsilon^3)x + 1 + 5\epsilon + 8\epsilon^2 + 5\epsilon^3 + \epsilon^4 = 0$$

finding the three terms in the approximation for each root, $x = x_0 + \epsilon^\alpha x_\alpha + \epsilon^{2\alpha} x_{2\alpha}$, and determining α along the way.

4. Develop three terms of the perturbation solutions to the real roots of

$$(x^3 + 2x^2 + x)e^{-x} = \epsilon,$$

identifying the scalings in the expansion sequence $\delta_0(\epsilon)x_0 + \delta_1(\epsilon)x_1 + \delta_2(\epsilon)x_2 + ...$

Eigenproblems and regularly perturbed differential equations

1. Find the corrections to the leading-order eigenvalues of the matrix problem

$$\begin{pmatrix} -1 & \alpha \\ \beta & -1 \end{pmatrix} \mathbf{x} = \lambda \mathbf{x} + \epsilon \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \mathbf{x},$$

for all possible values of the real parameters α and β.

2. By posing $\lambda = \lambda_0 + ...$ and $y = \epsilon y_1(x) + ...$, where ϵ corresponds to the small maximum amplitude of $\epsilon y_1(x)$ (*i.e.* y_1 has unit amplitude), find the (nontrivial) nonlinear correction to the leading-order eigenvalues λ_0 of the differential equation,

$$y'' + \lambda y + y^2 = 0,$$

with $y(0) = y(\pi) = 0$.

3. Normal modes of a slightly mis-shapen membrane

Normal-mode solutions to the wave equation $\nabla^2 \phi = \phi_{tt}$ take the form $\phi(x, y, t) = \Phi(x, y) \cos(\omega t)$ and therefore satisfy

$$\Phi_{xx} + \Phi_{yy} = -\omega^2 \Phi,$$

where subscripts denote partial derivatives. Consider a slightly mis-shapen membrane covering the domain,

$$0 \le x \le \pi, \quad \epsilon(\pi - |2x - \pi|) \le y \le \pi - \epsilon\sigma(\pi - |2x - \pi|),$$

with $\Phi = 0$ on the boundary, where $\sigma = \pm 1$. Show that $\Phi = \sin nx \sin my$ is a leading-order eigenfunction, with n and m integers. Find the corresponding eigenvalue ω. Calculate the $O(\epsilon)$ correction to the eigenvalue for (a) $n = m = 1$ and $\sigma = +1$. Comment on (but do not solve explicitly) the cases (b) $n = m = 1$ and $\sigma = -1$, and (c) $m = 2$, $n = 1$ and $\sigma = +1$.

Integrals

1. Use the method of repeated integration by parts or rescaling to obtain four terms in the asymptotic approximation to the integral,

$$\int_x^\infty \frac{(1+t^3)}{t^3} e^{-t} dt,$$

for $x \to 0$.

2. Find the leading-order behaviour for $x \gg 1$ of

$$(a) \int_0^\infty e^{xt(5-t^4)} \sin t \, dt \qquad (b) \int_0^\infty \exp(-t - x\sin t^2) \, dt \qquad (c) \int_0^{\frac{1}{2}\pi} \sin t \, e^{-x \sin^3 t} dt$$

3. Evaluate the first two terms as $\epsilon \to 0$ of

$$\int_0^\infty \frac{dx}{(\epsilon^2 + x^2)^{\alpha/2}(1+x)},$$

for $\alpha = \frac{1}{2}$, 2 and 1, if

$$C(\alpha) = \int_0^\infty \left[\frac{1}{(1+u^2)^{\alpha/2}} - \frac{1}{(1+u)^\alpha} \right] du.$$

4. Evaluate the first two terms as m approaches unity from below of

$$\int_0^{\pi/2} \frac{\sin^2\theta}{(1 - m^2 \sin^2\theta)^{1/2}} d\theta$$

5. Evaluate the first two terms as $\epsilon \to 0$ of

$$\mathcal{N}(z) = \int_z^1 \frac{dx}{\sqrt{x^3 + \epsilon}}$$

where $0 \le z < 1$.

Matched asymptotic expansions

1. Consider

$$\epsilon y'' + y' + y + \epsilon^2 y^2 = 0, \qquad \text{in } 0 \le x \le 1,$$

with $y(0) = 0$ and $y(1) = e^{-1}$. Find three terms of the outer solution, applying only the boundary condition at $x = 1$. Next find three terms in an inner approximation for the boundary layer near $x = 0$ applying the boundary condition at $x = 0$. Determine the constants of integration by matching (a) over an intermediate region, and (b) using van Dyke's rule with $P = Q = 2$. Compute the composite approximation, $C_{2,2}y$.

2. The function $y(x)$ satisfies

$$\epsilon x^p y'' + y' + y = 0, \qquad \text{in } 0 \le x \le 1,$$

for $p < 1$, $y(0) = 0$ and $y(1) = 1$. First find the rescaling for the boundary layer near $x = 0$, and obtain the leading order inner approximation. Then find the leading order outer approximation and match the two approximations.

3. Calculate two terms of the outer solution of

$$x^2 y' = \epsilon[(1+\epsilon)x^2 y^2 - x^2 + y^2] \qquad \text{in } 0 \le x \le 1,$$

with $y(1) = 1$. Locate the non-uniformity of the asymptoticness and hence the rescaling for an inner region. Thence find two terms for this inner solution. Is there another boundary layer nested inside the inner region?

4. The function $f(x)$ satisfies

$$f_{xx} - \frac{1}{2x^{3/2}} \epsilon f f_x = 0, \qquad f(0) = 1, \qquad f(1) = 0.$$

Obtain an asymptotic expansion for f at fixed x and $\epsilon \to 0$ in the asymptotic sequence, 1, ϵ, $\epsilon^2 \log(1/\epsilon)$, ϵ^2. Then find an expansion for f at fixed $\xi = x\epsilon^{-\alpha}$ as $\epsilon \to 0$ for some α (that you should determine), in the sequence 1, ϵ^2. Match these expansions.

Multiple Scales

1. Find equations, valid for times of $O(\epsilon^{-1})$, for the amplitude and phase of the leading-order solution to

$$\ddot{x} + 4x = \epsilon f(x, \dot{x}, t), \qquad x(0) = 0, \quad \dot{x}(0) = 1,$$

for

$$(a)\ f = -x^2\dot{x}, \qquad (b)\ f = x^5, \qquad (c)\ f = \dot{x}\sin 4t.$$

Solve these equations.

2. Find the leading-order approximation for times of order ϵ^{-1} to

$$\ddot{x} + x = y, \qquad \dot{y} = \epsilon(xy - 2y^2), \qquad x(0) = 1, \quad \dot{x}(0) = 0, \quad y(0) = 1,$$

3. Obtain an asymptotic approximation for x to order one, which is valid for $t = O(\epsilon^{-1})$, when

$$\ddot{x} + x + \epsilon|\dot{x}|(\dot{x} + x) = 0, \quad x(0) = 1, \quad \dot{x}(0) = 0.$$

4. Argue that the leading-order solution of the ODE,

$$\ddot{y} + (1 + \epsilon^2 a_2 - 2\epsilon\cos t + \epsilon^2 y^2)y = 0,$$

depends on the two timescales $(\tau, T) = (t, \epsilon^2 t)$. Hence obtain equations for the amplitudes, $A(T)$ and $B(T)$, in $y \sim A\cos\tau + B\sin\tau + O(\epsilon)$.

Bonus: What happens if we start the system off with $(A, B) = (0.01, 0)$?