Asymptotics final

Answer as much as you can.

1(a). For $\delta \ll 1$, find two terms in the asymptotic approximation of the three roots of the polynomial $x^3 + x^2 - \delta$.
1(b). For $z \gg 1$ and $z^{-2/3} \ll \delta \ll 1$, use Laplace's method to find the leading-order approximation to the integral,

$$\int_0^\infty x^2 e^{-zf(x)} dx, \qquad f(x) = \frac{1}{4}x^4 + \frac{1}{3}x^3 - \delta x.$$

Be as explicit as you can in your answer and estimate the error in your approximation.

2(**a**). Using repeated integration by parts or otherwise, find the coefficients of the $\xi \gg 1$ asymptotic approximation to the integral,

$$\int_0^\xi e^{-t^{-1}} dt \sim a_0\, \xi + a_1 \log \xi + a_2 + ...,$$

noting that Euler's constant is

$$\gamma = -\int_0^\infty e^{-x} \log x \; dx.$$

2(b). The function $f(r)$ satisfies the equation,

$$x^2 \frac{d^2 f}{dx^2} - \epsilon f \frac{df}{dx} = 0,$$

in $x \leq 1$, with $\epsilon > 0$ and the boundary conditions, $f = 0$ on $x = 1$ and $f \to 1$ as $x \to 0$. Obtain three terms of an asymptotic expansion for f at fixed x as $\epsilon \to 0$. Then find a three-term expansion for f at fixed $\xi = \epsilon^\alpha x$ as $\epsilon \to 0$, where you should determine α. Match these expressions.

3. Using multiple scales, find the leading-order asymptotic approximation, valid for $t = O(\epsilon^{-1})$ to the solution of the equations,

$$\dot{x} = y, \qquad \dot{y} + x = z, \qquad \dot{z} = \epsilon(xz - 3z^2), \qquad x(0) = y(0) = 0, \quad z(0) = 1,$$

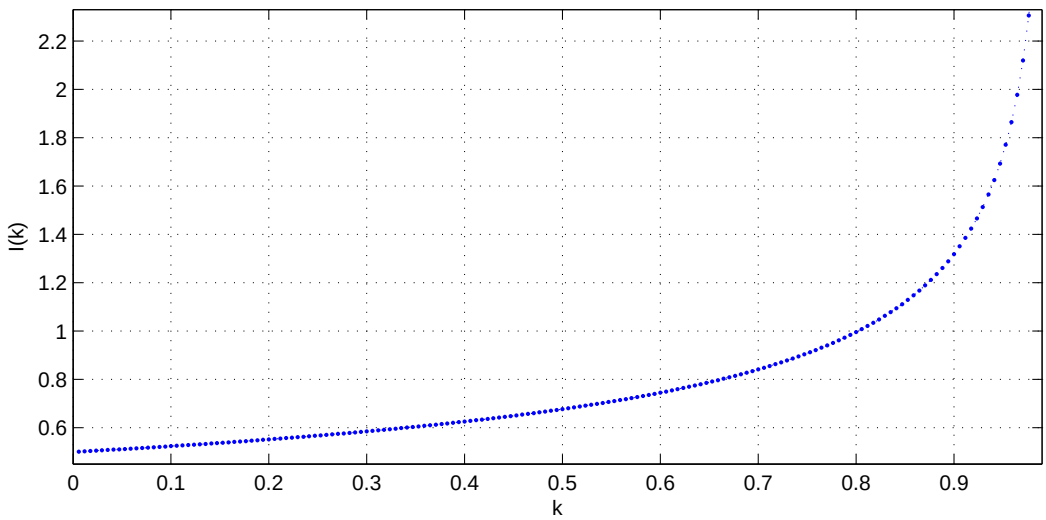
4(a). Consider the eigenvalue problem,

$$y'' + \lambda(a + x)(1 - x)y = 0, \qquad 0 \leq x \leq 2, \qquad y(0) = y(2) = 0\ .$$

where $a > 0$ is a parameter. Using the WKB method, provide an approximation for the eigenvalue, λ, comparing your result with the first five solutions obtained numerically for $a = 1$: $|\lambda| \approx 4.93$, 9.01, 26.28, 49.0 and 64.82. *You may quote the WKB formulae for* $y'' + f(x)y = 0$,

$$y \sim \omega^{-1/2}(a \cos\theta + b \sin\theta), \quad \omega^2 = f > 0, \quad \theta = \left| \int_{x_*}^x \omega(x')\; dx' \right|,$$

$$y \sim \Omega^{-1/2}(Ae^{\Phi} + Be^{-\Phi}), \quad \Omega^2 = -f > 0, \quad \Phi = \left| \int_{x_*}^x \Omega(x')\; dx' \right|,$$



with the connections $a = \sqrt{2}B + A/\sqrt{2}$ *and* $b = \sqrt{2}B - A/\sqrt{2}$*, if* x_* *is a turning point.*

(b) Provide an analogous result for $a = 0$, noting that

$$w'' + \lambda^2 x^{p-2} w = 0 \qquad \Longrightarrow \qquad w(x) = \sqrt{x}\, \mathcal{C}_{1/p}(2\lambda x^{p/2}/p),$$

where $\mathcal{C}_\nu(z)$ is a Bessel function of order ν. Compare your predictions for the first five eigenvalues with those determined numerically ($\lambda \approx 7.94$, 28.86, 42.64, 105.53 and 178.12.). How much of a difference does it make to the approximation of λ when the limit $x \to 0$ is properly dealt with?

5. (a) For the integral,

$$I(k) = \int_0^1 \frac{x dx}{(1 - kx^4)^{4/3}},$$

find the general term in an expansion for $k \ll 1$.

(b) Next, from the $k \ll 1$ series solution find the nearest singularity to the origin k_0 and its type, α. Returning to the integral, make a second approximation about the singularity, obtaining the first term of the approximation $I(k) \sim A(k_0 - k)^\alpha$.

(c) Use multiplicative and additive extraction to remove the nearest singularity in the small-k approximation, keeping terms upto and including order k^2.

(d) For $n = 0$ to 7, compute $S_n(k)$, the $(n+1)-$term approximation of $I(k)$. Use the Shanks transform to generate five improved approximations of $I(k)$. Iterate the Shanks transform to find even better approximations.

(e) Construct the (2,2) Padé approximant from the $k \ll 1$ series solution. At what value of k does this approximant place the singularity?

For $0 \le k \le 1$, sketch $S_7(k)$ against k and compare the result with the numerical computation shown in the figure; **use a printout of the figure if needed!** To the plot, add $I(k) \sim A(k_0 - k)^\alpha$, the two improved series from (c), the best approximation from the Shanks transforms in (d), and the (2,2) Padé approximant from (e). For a numerical comparison, compare all these results with the numerically determined value, $I(0.9) \approx 1.3184$.