

Rolling on custard

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We report a striking traveling wave instability in low Reynolds number flows of aqueous concentrated suspensions of corn starch. These experimental observations are at odds with theoretical predictions for simple shear thickening flows which are supposedly stable at low Reynolds number. A distinct threshold in Reynolds number is observed above which the finite amplitude disturbances persist and otherwise decay.

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Aqueous suspensions of corn starch show several remarkable features: as many know from observations in the kitchen, or from dining hall trials in English schools. These result from the rheology of the suspension which is decidedly non-Newtonian [1]. We highlight two salient properties of the material. First, on stirring or deforming in other ways, the fluid clearly possesses a resistance to flow that *increases* with the rate of flow. That is, there is an apparent shear-thickening effect: when sufficiently vigorous shear forcing is applied at a free surface, the surface appears to solidify and even fracture. Second, the material appears to possess a relaxation time: the concentrate reacts differently to impulsive or sudden loading differently than it does to a slow forcing. For example, a probe, gently applied, slides easily into the material, whilst rapid insertion encounters considerable resistance. Alternatively a compressed handful of corn starch suspension, when released, appears dry and temporarily solid: it can be crumbled and easily breaks. However, in the absence of external forcing, it appears to melt and flow away. The timescale over which this change occurs indicates a relaxation timescale, such as exists for viscoelastic fluids. Nevertheless, the material does not appear to exhibit other classical viscoelastic rheological features, such as rod climbing of a rotating spindle, open siphoning, or other normal stress effects.

Although a commonly encountered material, to our knowledge there are few fluid dynamical experiments that have been performed with cornstarch suspensions. We mention only two previous ones: First, Simpson [2] reported wave formation on a flowing layer of custard powder in water. His objective was to offer the flow as a laboratory analogue for a debris flow, although whether the custard suspension mimics properties of that geophysical material is certainly debatable. Nevertheless, Simpson reported an interesting fluid phenomenon which has, thus far, gone unexplained theoretically and unquantified experimentally, and our purpose in the present letter is to record efforts in this particular direction.

Second, Merkt *et al.* [1] have recently performed the

Faraday experiment with corn starch suspensions, uncovering some novel flow dynamics. In particular, they found that persistent “holes” appear in the vibrated fluid layer, with surrounding elevated collars, which they attribute to the shear-thickening property of the material.

The current article describes a new suite of experiments on the flow dynamics of cornstarch. Specifically, we have performed a laboratory study on wave formation on a layer of corn starch flowing down an inclined plane. Instabilities on falling fluid films of water are an everyday phenomena, being commonly seen on gutters and windows on rainy days, and have a well-established theoretical rationalization in terms of the linear instability of a uniform flow [3, 4]. This is the so-called Kapitza problem, for which theory predicts that instabilities arise when the Reynolds number, $Re = UH/\nu$, based on the surface flow speed, U , and depth, H , exceeds a critical value of order unity (ν is the kinematic viscosity). The instability prompts the growth of what are commonly referred to as “roll waves” [5, 6], which resemble propagating hydraulic jumps. The surprising property of the corn-starch suspensions examined here is that similar waves arise on a falling film, but at Reynolds numbers far below the critical value appropriate for a Newtonian fluid. These are the waves first reported by Simpson [2], but in a suspension of custard (which is cornstarch with flavouring). We proceed by reporting the results of an experimental investigation of this phenomenon, indicating how it defies current theoretical explanation, and discussing a number of physical mechanisms that might be responsible.

Experiments. Experiments were conducted on the flow of a concentrated solution of corn starch in water down a constant incline. The laboratory set-up involved a rectangular chute, 10 cm wide and 1.5 or more meters long, fed upstream from a reservoir whose level was kept roughly constant. By varying the angle of inclination we were able to vary flow speed and depth. Typical fluid thickness were of the order of 0.5-1 cm; flow speeds were in the 1-10cm/sec range. We also varied the concentration of the cornstarch in solution; suspensions slightly in

excess of 1 part cornstarch to 1 part water, by weight, were those that showed the phenomena described below.

If the fluid were Newtonian, the force of gravity down the plane, associated with the gravitational acceleration $g \sin \theta$ (where θ is the angle of the plane and $g \approx 9.81 \text{m/sec}^2$), would be balanced by that stemming from the viscous shear stress, νu_{zz} (orientating a two-dimensional Cartesian coordinate system so that $z = 0$ denotes the inclined plane, x points directly downslope, and (u, w) denote the velocity field). Very roughly, this signifies that

$$\frac{\nu U}{H^2} \sim g \sin \theta.$$

Hence, $Re = UH/\nu \sim U^2/(gH \sin \theta)$. For the roll wave typically seen on films of water, i.e. the Kapitza problem, the fluid is millimeters in depth and flows at centimeters per second on a shallow slope of perhaps 10 degrees, suggesting a Reynolds number of order 10 or more (comfortably above the theoretical prediction of $(5/4) \cot \theta$ [3, 4]). For our cornstarch solutions, on the other hand, we see waves at smaller Reynolds numbers of order 0.1.

A picture showing the developing roll waves is shown in figure 1(a) and (b). Naturally occurring perturbations apparent at the inlet seed growing, propagating disturbances that steepen into an unsteady train of waves. Near initiation, the waves are remarkably regular, and fairly evenly spaced, with a wavelength of a few centimeters. They quickly grow and reach relatively large amplitude (the heights of their crests can be a significant fraction of the fluid depth), and generally form with a range of strengths. As a result, the waves travel with different speeds, overtaking one another in their progression down the chute. The collision of two waves leads to their merger into a larger wave. Thus the wavetrain undergoes a process of coarsening, as illustrated in figure 1(c), which is a typical feature of many nonlinear wave propagation problems [7]. Surprisingly, both the initial wavelengths and the coarsened wavelengths further down the channel showed little dependence on the inclination of the chute.

Though showing a spread of speeds, the roll waves travel with velocities that typically exceed the flow speed by approximately 33-50 percent or so. The structure of the waves is also unusual: the fluid surface appears to be advected with the waves, creating a characteristic caterpillar-like action as the disturbances “roll” over the fluid layer ahead.

Lastly, by varying the inclination, we observe what appears to be a critical threshold in slope (that depends on concentration) below which small waves initiated at the top of the chute no longer grow or persist as they propagate, but decay to the base flow (see figure 1(d)). The critical angle is shown in figure 2 where it is plotted against concentration. Note that this critical angle is difficult to establish for two reasons. First, and least im-

portantly, the chute is not sufficiently long to state definitively whether a small disturbance grows or decays over the timescale of the experiment. Second, even though it is clear that small amplitude waves decay in the “stable” regime in the experiment, it appears that larger amplitude disturbances are able to persist and survive. In other words, it appears that there may be a finite-amplitude instability, which significantly complicates the identification of the threshold slope, especially given the large-amplitude perturbations encountered at the inlet (see below).

Further details and observations.

Jitter. In addition to the main wave generation process, the flow of the corn starch suspension exhibited another curious feature resembling a superimposed high-frequency jitter or flutter. More precisely, the large-scale flow of the material appeared to generate a high-frequency vibration in a manner reminiscent of the generation of acoustic waves by flow in a compressible fluid. The jitter could be seen clearly at the crests of the roll waves, where the caterpillar-like overturning motions vibrated with periods of order 0.1 seconds. However, simply pouring the material from one receptacle to another also excited the vibrations, to the degree that they could be easily felt if one were to hold one of the containers. This process was also responsible for creating substantial agitation in the feeding reservoir at the top of the chute, which in turn quite clearly helped to seed the roll waves themselves. An unappealing consequence was that the nucleation of fairly large-amplitude roll waves obscured the identification of the stability boundary.

Ageing. After repeating experiments over the course



FIG. 1: A typical experimental view of (a) developed roll waves, (b) a detailed view of a pair of waves, (c) merging waves at a downstream location, and (d) a stable flow.

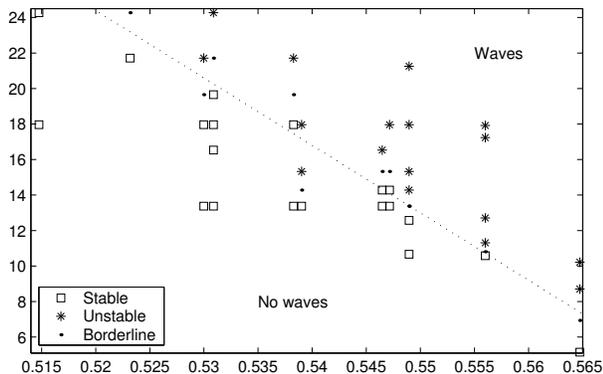


FIG. 2: The plot of slope versus concentration showing the threshold values. The stars show flows in which waves appeared to amplify as they propagated downstream, the squares represent flows in which the waves appeared to decay, and the dots show cases that seemed borderline between those two behaviours. The lines are a fit to the data.

of days with fresh or old material, it became clear that the properties of the corn starch suspensions were slowly evolving, presumably owing to some combination of evaporation of the suspending fluid and the swelling of the starch grains. For example, the material used in the experiment illustrated in figure 1 was fresh (prepared at most an hour before the experiment). Leaving the suspension for a period of a day or two allowed the cornstarch to separate from the solution, although it could be re-mixed to leave a suspension that appeared much the same as before. However, when the flow experiments were performed a second time, the roll waves that appeared were very different. In particular, the critical slope angle increased, the roll-wavelength was visibly smaller, the instability appeared to saturate at much lower amplitude and there was little sign of coarsening. Snapshots of the re-run experiment are shown in figure 3.

Theoretical background. One of the primary theoretical problems posed by the cornstarch suspension is characterizing its rheology. Existing studies suggest that the suspension can be shear thickening (and even thinning over some ranges of applied shear) and show a definite relaxation time [8], much as we anticipated on observational grounds. However, there is currently no accepted rheological model for this material. Instead, given its apparent behaviour, we outline known theoretical results for commonly used shear thickening or visco-elastic fluid models.

First, shear-thickening behaviour is captured in the standard power-law fluid model, for which the viscosity, ν , depends on the local deformation rate, $\dot{\gamma}$: $\nu = K\dot{\gamma}^{n-1}$, where K is a constant. Long-wave stability theory [7, 9] establishes that the critical Reynolds number for this flow, $Re = U^{n-2}H^n/K$, is given by $Re = Re_c \equiv \cot \theta (2n+3)/4$. (For $n = 1$ we recover the familiar New-



FIG. 3: Roll waves on fluid layers of aged material. The second panel shows an experiment in which the inclination was steeper than that shown in the first panel.

tonian result of Benjamin [3] and Yih [4].) Evidently, the critical threshold increases with n : shear thickening fluids (with $n > 1$) are *more* stable than Newtonian and shear thinning ones. Thus, the observed behaviour of the cornstarch suspension apparently cannot be rationalized in terms of a simple shear-thickening rheology. In fact, roll waves have been observed on mud flows (kaolin suspensions) in laboratory flumes [9, 10], and mud is usually thought to be shear thinning. The experimental observations suggest critical Reynolds numbers that are much higher than those observed with our cornstarch suspensions, and which are more consistent with the theoretical predictions based on power-law fluid model. Moreover, the observed wavelengths of muddy roll waves are much longer than those seen in the cornstarch suspension. Recent theory for shear thickening, viscoplastic fluids [11] suggests that yield stresses could substantially lower the critical Reynolds number. However, cornstarch suspensions show apparently negligible yield stresses, draining as they do from inclined surfaces over long timescales to films that are very thin indeed.

Second, long-wavelength stability theories have been presented for a number of archetypal viscoelastic fluids [12, 13]. In particular, for the Oldroyd B model, it has been shown that elasticity can be destabilizing: the critical Reynolds number is lowered by an amount proportional to the dimensionless polymer relaxation rate, the Deborah number. If the relaxation rate were of order seconds, and the shear rates were comparable (as they are for fluids of depth a centimeter and speed of centimeters

per second), this elasticity parameter is order one, and therefore important. Given that the observed, apparent relaxation rate of the cornstarch suspension is close to this apparent shear rate, there is a possibility that it is the relaxation of the material that is partly responsible for the diminution of the critical Reynolds number. However, as previously mentioned, the cornstarch suspension does not exhibit the classic characteristics of polymeric, viscoelastic fluids. Moreover, there has been some question over whether this type of viscoelastic instability could be observable [13].

Third, roll waves have lately been reported on flowing granular layers [14]. The theory used to rationalize this phenomenon is based on shallow fluid models akin to the St. Venant model of hydraulic engineering. Simple friction laws are used to model the granular fluid stresses, and there is some agreement between theory and observation. However, the flow speeds and depths again suggest effective Reynolds numbers that are well above those encountered in our cornstarch suspensions, and more in line with the critical values expected for viscous fluids.

Discussion. In summary, we are unable to rationalize convincingly the appearance of roll waves at such lower Reynolds number in cornstarch suspensions in terms of an existing, accepted rheological model. To provide an answer to this puzzle, more studies are required on this material, both at the microscopic and macroscopic levels. Microscopic studies are needed to understand what properties of cornstarch particles are responsible for its macroscopic behaviour particularly at high concentrations. Macroscopic studies are needed to better identify its rheology for theoretical modelling.

Because we have no good theoretical explanation, we are limited to speculating on what are the likeliest possibilities, and we close by offering our views on this.

Wave excitation at such low Reynolds numbers could be attributed to some kind of jamming phenomenon in a highly concentrated suspension [15–17]: a localized perturbation may jam particles together into a coherent structure that accelerates to collect and jam further particles into a growing mass. However, there does not currently appear to be much of a rational continuum formulation of this process to call upon. Related notions have appeared in other contexts, such as in thixotropic fluids where it has been suggested that particle interaction (gauged by some internal restructuring variable) can lead to a viscosity bifurcation and thence instability [18–20] In a sense, this behaviour results from what might be called “constitutive instability”, which is also a common concept in viscoelastic fluid mechanics [21, 22].

Another possibility is related to the idea that the suspension develops inhomogeneity: suspended particles are known to migrate in solute, leaving regions of high shear or boundary layers [23]. Effective slip in the particle-depleted layers, both internal and at a wall, can result from this (a phenomenon known to plauge rheologists). A

similar phenomenon arises in media like wet sand, where shear perturbs the solid matrix from its state of optimal packing, thus increasing the fluid fraction at the base of the layer. Either way, a more viscous, particle-rich layer develops over a less viscous layer. A stick-slip instability is then an obvious possible consequence, as is interfacial instability due to the effective, abrupt change in viscosity [24]. Unfortunately, although such explanations are appealing, it remains unclear whether the form of the instability could resemble our roll waves.

For the observed high frequency jitter, it is notable that avalanche and large scale flows of so-called singing or booming sand and fluidized granular media produce acoustic signals through rapid surface and internal vibrations of the grains [25, 26]. Elasto-plastic models also suffer what is referred to as “flutter” instability [27], which takes the form of unstable propagating waves with speeds near those expected for shear and compressional waves (*i.e.* relatively high speed).

Whatever the physical origin of the phenomena we have observed, we hope that the study motivates further study of this curious material.

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