Justification of PDE solution, $u = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin nx$

the solution is defined for

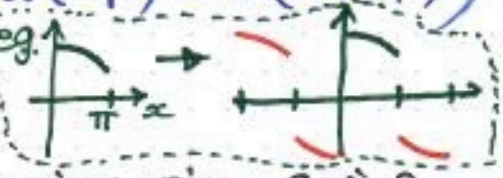
$0 < x < \pi$ by the PDE

and at $0, \pi$ by the BCs.

Demand that $u(x,t) = -u(-x,t)$ for $-\pi < x < 0$

& then be 2π-periodic $\left(u(x,t) = u(x+2\pi, t) \right)$

This is the "Odd, periodic extension" of $u(x,t)$.



Odd, periodic functions have a Fourier sine series

$$(a_0 = a_n = 0)$$

[Because $\int_{-L}^{L} f(x)\,dx = \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\,dx = 0$

owing to symmetry properties]

This particular extension of u (& $f(x) = u(x,0)$) therefore guarantees that we may represent the solution this way, regardless of the form of the IC (as long as it is not singular)

Alternative strategy: Expansion by eigenfunctions (or basis functions).

Take the odd periodic extension of $u(x,t)$ at the outset, then represent it as a sine series:

$$u(x,t) = \sum_{n=1}^{\infty} B_n(t) \sin nx$$

↖ the F.S. is in x!

$B_n(t)$ ↑ accommodates t-dependence.

Plug into PDE, match up coeffs of $\sin nx \Rightarrow$

$$\boxed{\dot{B}_n = -n^2 B_n}$$

* B.C.'s already taken care of

* I.C. gives $B_n(0) = b_n$

$\longrightarrow B_n = b_n e^{-n^2 t}$ & the PDE sol. obtained before.