Integral form of heat eq.

$$\int_a^b \rho c_p \frac{\partial T}{\partial t}\,dx = \left[k\frac{\partial T}{\partial x}\right]_a^b + \int_a^b S\,dx$$

But $\int_a^b I(x)\,dx = 0 \Rightarrow I(x) = 0$ if a & b are arbitrary.

$$\Rightarrow \rho c_p \frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} + S \qquad \text{(if } k \text{ is const.)}$$

→ differential form.

Scaling: often we can remove constant coeffs. by rescaling the independent variables. (eg. $x = \alpha\hat{x}$, $t = \beta\hat{t}$)

This eliminates distracting constants (which are physically important as one must reconcile dimensional units, but not for subsequent math manipulation).

So, our equat. is basically $\boxed{u_t = u_{xx} + q}$

$\boxed{\text{Solution by Separation of Variables of}}$ $u_t = u_{xx}$ (set $q=0$ for now), $u=0$ at $x=0,\pi$, $u=f(x)$ at $t=0$.

Let $u = X(x)T(t)$ & divide by XT ... $\longrightarrow \underbrace{\frac{T'}{T}}_{\text{fun}(t)} = \underbrace{\frac{X''}{X}}_{\text{fun}(x)}$ (prime is derivative w.r.t. argument)

→ only true if both equal a const.

anticipate decay for diffusion

$\therefore T' = -\lambda T,\ X'' = -\lambda X$ $\quad \lambda \equiv$ separation const.

$\Rightarrow u = (a\cos\sqrt{\lambda}x + b\sin\sqrt{\lambda}x)e^{-\lambda t}$ But $u=0$ at $x=0$ $\Rightarrow a = 0$.

Then $u=0$ at $x=\pi \Rightarrow \lambda = n^2$ for $n=1,2,3,\ldots$ ← wrong sign choice for λ would fail here.

Gen. Sol. $\quad u(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx)\,e^{-n^2 t}$

Last, $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ is satisfied if $b_n = \frac{2}{\pi}\int_0^{\pi} f\sin nx\,dx$