Chemical waste released into a river from a factory at $x = 0$ has concentration $u(x,t)$. The amount of the chemical released is $u(0,t) = C(t)$. In the river, the chemical is advected in $x > 0$ and depleted according to

$$u_t + v(x)u_x = -\mu u, \tag{1}$$

where $v(x)$ is the river flow speed and $\mu > 0$ is the (constant) depletion rate. The river was clean initially, $u(x,0) = 0$, and the total amount of pollutant in it is

$$U(t) = \int_0^\infty u(x,t)dx.$$

(i) Provide a formula for $\overline{u}(x,s) = \mathcal{L}\{u(x,t)\}$ in terms of $\overline{C}(s) = \mathcal{L}\{C(t)\}$ and

$$T(x) = \int_0^x \frac{d\hat{x}}{v(\hat{x})},$$

and hence write down a solution for $u(x,t)$ in terms of $C(t)$ and $T(x)$.

By monitoring the river, the factory attempts to control the environmental impact of the chemical release by demanding that $C(t) = P(t) - U(t)$, for some base production function $P(t) > 0$.

(ii) Provide a formula for $\overline{C}(s)$.

Now take $v(x) = \frac{1}{2} + x$.

(ii) Given that $\mathcal{L}\{\int_0^t f(t-\tau)g(\tau)d\tau\} = \overline{f}(s)\overline{g}(s)$, write down the solution for $C(t)$ as a convolution integral.

For the last three parts, consider the case that $P(t) = \textit{constant}$:

(iv) Provide explicit solutions for $\overline{C}(s)$, $C(t)$ and $u(x,t)$.

(v) Establish the limits of $C(t)$ and $u(x,t)$ for $t \to \infty$, and compare these limits with the steady solution $u = w(x)$ to (1).

(vi) Compare the limits found in *(v)* with the limits of $s\overline{C}(s)$ and $s\overline{u}(x,s)$ for $s \to 0$, rationalizing from the definition of the Laplace transform why they are related, or if not, why they are not.

Solution:

Laplace transforming the PDE:

$$(s+\mu)\overline{u} + v\overline{u}_x = 0, \qquad \text{or} \qquad \overline{u} = \overline{C}(s)e^{-(s+\mu)T(x)},$$

given that $\overline{u}(0,s) = \overline{C}(s)$ and from the definition of $T(x)$. The inverse transform, using the second shifting theorem, is

$$u(x,t) = e^{-\mu T(x)}C(t - T(x))H(t - T(x)).$$

Laplace transforming $U(t)$ and $C(t) = P(t) - U(t)$, then plugging in the solution for $\overline{u}(x,s)$ gives

$$\overline{C}(s) = \overline{P}(s)\left[1 + \int_0^\infty e^{-(s+\mu)T(x)}dx\right]^{-1}.$$

For $v(x) = \frac{1}{2} + x$, $T = \ln(1+2x)$ and we find

$$\overline{C}(s) = \frac{\overline{P}(s+\mu-1)}{(s+\mu-\frac{1}{2})} = \overline{P} - \frac{\overline{P}}{2(s+\mu-\frac{1}{2})} \qquad \Longrightarrow \qquad C(t) = P(t) - \tfrac{1}{2}\int_0^t P(t-\tau)e^{-(\mu-\frac{1}{2})\tau}d\tau.$$

For the case $P = \textit{constant}$, we arrive at

$$\overline{C}(s) = \frac{P(s+\mu-1)}{s(s+\mu-\frac{1}{2})}, \qquad C(t) = \left(\frac{\mu - 1 + \frac{1}{2}e^{-(\mu-\frac{1}{2})t}}{\mu - \frac{1}{2}}\right)P$$

and

$$u(x,t) = \frac{P}{\mu - \frac{1}{2}} \left[(\mu-1)(1+2x)^{-\mu} + \tfrac{1}{2} e^{-(\mu-\frac{1}{2})t} (1+2x)^{-\frac{1}{2}} \right] H(t - \ln(1+2x)).$$

In the special case that $\mu = \frac{1}{2}$,

$$C = P(1 - \tfrac{1}{2}t) \qquad \& \qquad u = P(1+2x)^{-\frac{1}{2}} \{1 - \tfrac{1}{2}[t - \ln(1+2x)]\} H(t - \ln(1+2x)).$$

For $\mu \leq \frac{1}{2}$, the solutions diverge as $t \to \infty$, implying no limits. Admittedly, this result is suspicious: the concentration first becomes negative before diverging, implying that the model breaks down (concentration should be positive), when $C(t)$ reaches zero.

When $\mu > \frac{1}{2}$, on the other hand,

$$C \to \frac{P(\mu-1)}{\mu - \frac{1}{2}} \qquad \& \qquad u \to \frac{P(\mu-1)(1+2x)^{-\mu}}{\mu - \frac{1}{2}}.$$

The steady solution satisfies $vw_x = -\mu w$. As long as $\mu > 1$, this implies

$$w = C(1+2x)^{-\mu} \qquad \& \qquad C = P \left[1 + \int_0^\infty (1+2x)^{-\mu} dx \right]^{-1} = \frac{P(\mu-1)}{\mu - \frac{1}{2}};$$

there is no steady state for $\mu \leq 1$. The limits of u and C therefore agree with the steady solution, but only when $\mu > 1$, a curious result originating from the convergence of the spatial integral in $U(t)$ with and without time-dependence.

The limits for $\mu > \frac{1}{2}$ are precisely equal to the limits of $s\overline{C}(s)$ and $s\overline{u}(x,s)$ for $s \to 0$. Denoting f_∞ as the limit of $f(t)$ for $t \to \infty$, and assuming this to be finite, we observe that

$$s\overline{f}(s) - f(0) = \int_0^\infty e^{-st} \frac{df}{dt} dt \to f_\infty - f(0) \quad \text{for } s \to 0,$$

establishing the origin of the coincidence. If f_∞ diverges, then we cannot take the limit $s \to 0$ to establish the result, avoiding any coincidence.