e.g. $u_{xx}+u_{yy}=0$

$-\infty<x<\infty \qquad u(x,0)=f(x)$

$0\le y<\infty$

FT in x ... $\qquad \hat{u}(k,y)=\mathcal{F}\{u(x,y)\}$

PDE $\rightarrow \quad -k^2\hat{u}+\hat{u}_{yy}=0$

BC $\rightarrow \quad \hat{u}(k,0)=\hat{f}(k)$

Hence $\quad \hat{u}(k,y)=\hat{f}(k)e^{-|k|y}$

since $\hat{u}\to 0$ for $y\to\infty$.

Transform table

$f(x)$	$\hat{f}(k)$
$e^{-a\|x\|}$	$\dfrac{2a}{a^2+k^2}$
$\dfrac{a}{\pi(a^2+x^2)}$	$e^{-a\|k\|}$
e^{-ax^2}	$\sqrt{\frac{\pi}{a}}\,e^{-k^2/4a}$
$f\circ g$	$\hat{f}\hat{g}$
$\delta(x-a)$	e^{-ika}
$\delta(x)$	1

Thus, $\quad u(x,y)=\displaystyle\int_{-\infty}^{\infty} f(\hat{x})\,g(x-\hat{x})\,d\hat{x}$

where $g(x)$ corresponds to $\hat{g}(k)=e^{-|k|y}$

& $f\circ g$ is a convolution integral.

$$f\circ g=\int_{-\infty}^{\infty}f(\hat{x})g(x-\hat{x})d\hat{x} \xrightarrow{FT} \mathcal{F}\{f\circ g\}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-ikx}f(\hat{x})g(x-\hat{x})\cdot dx\,d\hat{x} \; *$$

$dx\to d\check{x}$ (limits unchanged)

$\hat{x}+\check{x} \qquad \check{x}$

Put $x-\hat{x}=\check{x}$

$$\rightarrow \mathcal{F}\{f\circ g\}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-ik\hat{x}}\,e^{-ik\check{x}}\,f(\hat{x})\,g(\check{x})\,d\hat{x}\,d\check{x}$$

$$\equiv \hat{f}\,\hat{g}$$

But $g(x)=\dfrac{y}{\pi(y^2+x^2)}$ from table

$$\therefore\; u(x,y)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(\hat{x})\,\frac{y\,d\hat{x}}{y^2+(x-\hat{x})^2}$$

eg. $f(x)=\begin{cases}1 & \text{for } 0<x<1\\ 0 & \text{otherwise}\end{cases} \Rightarrow u=\dfrac{1}{\pi}\tan^{-1}\left(\dfrac{1-x}{y}\right)+\dfrac{1}{\pi}\tan^{-1}\left(\dfrac{x}{y}\right)$

δ-functions: have property $\displaystyle\int_{-\infty}^{\infty}\delta(x-a)F(x)dx=F(a)$

Hence $\mathcal{F}\{\delta(x-a)\}=e^{-ika}$

$\delta(x)$ is an example of a generalized func — makes sense mathematically only when one multiplies by a test func (F) & integrates.