Fourier Transforms

Given $f(x)$, with $f \to 0$ for $x \to \pm\infty$

$$\hat{f}(k) = \mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx$$

notation for F.T. Definition

k is a parameter for integral → gives new func of k.

Using an integration by parts (& given $f \to 0$ for $x \to \pm\infty$)

$$\mathcal{F}\{df/dx\} = ik\hat{f}.$$

transforms a derivative into an algebraic factor!

Inverse Transform:

$$f(x) = \mathcal{F}^{-1}\{\hat{f}(k)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\hat{f}(k)\,dk$$

notation

similar integral!

For $u(x,t)$: $\hat{u}(k,t) = \mathcal{F}\{u(x,t)\}$

(FT in x not t)

$$\mathcal{F}\{u_x\} = ik\hat{u}, \quad \mathcal{F}\{u_{xx}\} = -k^2\hat{u}$$

$$\mathcal{F}\{u_t\} = \hat{u}_t$$

(t-derivative slips out of x-integral)

provided $u \to 0$ for $x \to \pm\infty$.

Apply to: $u_t = u_{xx}$, $u(x,0) = f(x)$, $u, f \to 0$ for $x \to \pm\infty$.

FT of $\begin{cases} \text{PDE} \to \hat{u}_t = -k^2\hat{u} \\ \text{IC} \to \hat{u}(k,0) = \hat{f}(k) \end{cases}$ turns PDE into ODE!

$$\Rightarrow \hat{u}(k,t) = \hat{f}(k)e^{-k^2 t} \to u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f} e^{ikx-k^2t}\,dk$$

(reduced PDE sol. to 2 integrals)

eg. $f(x) = e^{-a|x|} \longrightarrow \hat{f}(k) = \dfrac{2a}{a^2+k^2}$ using FT

$\hat{f}(k) = e^{-a|k|} \longrightarrow f(x) = \dfrac{a}{\pi(a^2+x^2)}$ using inverse FT

$(a>0)$

$$\hat{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx \quad \text{(def.)}$$

For each known transform pair we get another for free

switch k & x

$$\hat{f}(x) = \int_{-\infty}^{\infty} e^{-ikx} f(k)\,dk$$

cf. def. of inverse FT

$\therefore$ if $\boxed{g(x) = \hat{f}(x) \quad \hat{g}(k) = 2\pi f(-k)}$

$$k \to -\tilde{k} \longrightarrow \hat{f}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\tilde{k}x}\, 2\pi f(-\tilde{k})\,d\tilde{k}$$

reciprocity