Associated Leg. functions

The ODE $\quad \frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + \lambda y - \frac{m^2 y}{1-x^2} = 0$

has solution $\quad C_n^m P_n^m(x)$, $\lambda = n(n+1)$ with $n = 0,1,2,\dots$ as before.

arb. const.

$$P_n^m \equiv (1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x)\,(-1)^m \quad \text{"Associated Legendre funk."}$$

N.B. we had an earlier convention that the SL eigenfunctions $\{\lambda_n, y_n\}$ started with $n=1$; here we're starting with $n=0$.
(because $\lambda \equiv n(n+1)$ for $n=0,1,2,\dots$)

Back to PDE:

$$u(\rho,\theta,\varphi) \rightarrow R(\rho)\,\Theta(\theta)\,\Phi(\varphi)$$

$\Phi'' + m^2\Phi = 0$, FS in φ, $m = 0,1,2,\dots$

gave the Leg. diff eqs. with $x = \cos\theta$, sols. $P_n^m(\cos\theta)$

ρ^ν or $\rho^{-1-\nu}$ with $\lambda = \nu(1+\nu)$, so $\nu \equiv n$!
And regularity demands ρ^n.

Gen. Sol. is $\therefore$

$$u = \sum_{n=0}^{\infty}\left[\frac{1}{2}a_{0n}P_n(x) + \sum_{m=1}^{\infty}\left(a_{mn}\cos m\varphi + b_{nm}\sin m\varphi\right)P_n^m(x)\right]$$

Boundary condition: $u = F(\theta,\varphi)$ at $\rho = 1$.

Use a F.S. to represent F:

$$F(\theta,\varphi) = \frac{1}{2}A_0(\theta) + \sum_{m=1}^{\infty}\left[A_m(\theta)\cos m\varphi + B_m(\theta)\sin m\varphi\right]$$

where

$$\left[A_0, A_m, B_m\right] = \frac{1}{\pi}\int_{-\pi}^{\pi}F(\theta,\varphi)\left[1, \cos m\varphi, \sin m\varphi\right]d\varphi$$

Match up the F.S. terms:

$$A_0 = \sum_{n=0}^{\infty}a_{0n}P_n(x)$$

$$A_m = \sum_{n=0}^{\infty}a_{mn}P_n^m(x)$$

$$B_m = \sum_{n=0}^{\infty}b_{mn}P_n^m(x)$$

Requires

$$a_{0n} = \int_{-1}^{1}A_0 P_n(x)\,dx \Big/ \int_{-1}^{1}P_n^2\,dx$$

$$\begin{Bmatrix} a_{mn} \\ b_{mn} \end{Bmatrix} = \int_{-1}^{1}\begin{Bmatrix} A_m \\ B_m \end{Bmatrix}P_n^m\,dx \Big/ \int_{-1}^{1}(P_n^m)^2\,dx$$