MATH 440/508
HW 1

Due September 20, 2019 (at the beginning of the class)

To be handed in

1. Show that the following are equivalent for an open subset Ω of the complex plane.
   (a) Any two points of Ω can be joined by a (piecewise smooth) curve. That is, for any $z_1, z_2 \in \Omega$, there exists a piecewise smooth curve in Ω that begins at z_1 and ends at z_2.
   (b) Any continuously differentiable function $h : \Omega \to \mathbb{R}$ such that $\nabla h = \left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}\right) = (0, 0)$ is constant. (Here we view Ω as a subset of $\mathbb{R}^2$ and h as a function of two real variables. By *continuously differentiable*, we mean that h is continuous in Ω, the partial derivatives $\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}$ exist and are continuous in Ω.)
   (c) Ω is connected. In other words, if U, V are disjoint open subsets such that $\Omega = U \cup V$, then either $U = \Omega$ or $V = \Omega$.
2. Chapter 1: Exercise 7
3. Chapter 1: Exercise 23
4. Let Ω denote the region

$$\Omega = \mathbb{C} \setminus (-\infty, 0] = \left\{ re^{i\theta} : r \in (0, \infty), \theta \in (-\pi, \pi) \right\}.$$

   Consider the function $F : \Omega \to \mathbb{C}$ defined as

$$F(z) = \log(r) + i\theta,$$

   where $z = re^{i\theta}$ such that $r \in (0, \infty), \theta \in (-\pi, \pi)$. Show that F is holomorphic in Ω and compute the derivative F'.
5. (a) Let Ω_1, Ω_2 be two open sets in $\mathbb{C}$ such that $\Omega_1 \cap \Omega_2$ is nonempty and connected. Let $f : \Omega_1 \cup \Omega_2 \to \mathbb{C}$ be holomorphic in $\Omega_1 \cup \Omega_2$. Assume that f has a primitive $F_1 : \Omega_1 \to \mathbb{C}$ in Ω_1 and f has a primitive $F_2 : \Omega_2 \to \mathbb{C}$ in Ω_2. Prove that f has a primitive $F : \Omega_1 \cup \Omega_2 \to \mathbb{C}$ in $\Omega_1 \cup \Omega_2$.

(b) Show by example that the hypothesis that $\Omega_1 \cap \Omega_2$ is connected is necessary.

Practice problems (not to be handed in):

- Chapter 1: Exercise 8, 13, 14, 16, 19, 26.

The following problems are meant to help you review MATH 320 (which is a prerequisite)

1. Let $\Omega \subset \mathbb{C}$ be a closed set. Let $z = \lim_{n\to\infty} z_n$, where $z_n \in \Omega$ for all $n \in \mathbb{N}$. Show that $z \in \Omega$.

2. Show that every convergent sequence in $\mathbb{C}$ is a Cauchy sequence.

3. Let (a_n) be a bounded sequence of real numbers. Show that $a = \limsup_{n\to\infty} a_n$ can be characterized as the unique real number satisfying the two properties:

   (i) If $\alpha < a$, then there are infinitely many $n \in \mathbb{N}$ with $a_n \geq \alpha$.

   (ii) If $\beta > a$, then there are only finitely many $n \in \mathbb{N}$ such that $a_n \geq \beta$.

   Recall that $\limsup_{n\to\infty} a_n$ is defined as $\lim_{n\to\infty} b_n$, where $b_n = \sup_{m\geq n} a_n$.

4. Let a_n be a sequence of complex numbers and let $s_n = \sum_{k=0}^{n} a_k$ denote the partial sums. If $\lim_{n\to\infty} s_n$ exists, then show that $\lim_{n\to\infty} a_n = 0$.

5. Show that the sequence

$$b_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \log n, \quad n \geq 1,$$

   is decreasing, while the sequence $a_n = b_n - \frac{1}{n}$ is increasing. Show that both the sequences converge to the same limit. (This limit is called the *Euler's constant*).

6. Let A, B be disjoints subsets of $\mathbb{C}$ such that A is compact and B is closed. Show that

$$\inf\{|a - b| : a \in A, b \in B\} > 0.$$