Continuously differentiable functions and Inverse function theorem

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We recall two equivalent definitions of the notions of continuously differentiable on n-dimensional Euclidean space $\mathbb{R}^n$. We use boldface letters to denote elements of $\mathbb{R}^n$. For $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we define its absolute value $|\mathbf{x}|$ as $\sqrt{\sum_{i=1}^n x_i^2}$. For $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n)$, we define the dot product $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^n x_i y_i$.

Differentiability can be defined using existence of a linear approximation as below.

d:def1 **Definition 0.1.** Let Ω be an open set in $\mathbb{R}^n$. Let $f : \Omega \to \mathbb{R}$ be a function. We say that f is differentiable at $\mathbf{x} \in \Omega$, if there exists $\mathbf{m} \in \mathbb{R}^n$ such that

$$\lim_{\mathbf{h} \to 0} \frac{|f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) - \mathbf{m} \cdot \mathbf{h}|}{|\mathbf{h}|} = 0.$$

The value $\mathbf{m}$ is called the *total derivative* of f at the point $\mathbf{x}$ and is denoted by $\mathbf{D}f(\mathbf{x})$. (If such an $\mathbf{m}$ exist then it is unique Rud [Rud, Theorem 9.12])

We say that a function $f : \Omega \to \mathbb{R}$ is *differentiable in* Ω if it is differentiable at all points $x \in \Omega$.

We say that $f : \Omega$ is *continuously differentiable in* Ω, if f is differentiable in Ω and the map $\mathbf{D}f : \Omega \to \mathbb{R}^n$ is continuous in Ω. (that is, for all $\mathbf{x} \in \Omega, \varepsilon > 0$, there exists $\delta > 0$ such that $|\mathbf{D}f(\mathbf{x}) - \mathbf{D}f(\mathbf{y})| < \delta$, whenever $\mathbf{y} \in \Omega$ is such that $|\mathbf{x} - \mathbf{y}| < \delta$).

Another equivalent definition of continuously differentiable functions uses partial derivatives.

d:def2 **Definition 0.2.** Let Ω be an open set in $\mathbb{R}^n$. Let $f : \Omega \to \mathbb{R}$ be a function.

We say that $f : \Omega \to \mathbb{R}$ is *continuously differentiable in* Ω, if f is continuous in Ω, the partial derivative $\frac{\partial f}{\partial x_i}(\mathbf{x})$ exists for all $i = 1, \ldots, n$ and for all $\mathbf{x} \in \Omega$, and the partial derivatives $\frac{\partial f}{\partial x_i} : \Omega \to \mathbb{R}$ are continuous for all $i = 1, \ldots, n$.

The following theorem is a special case of Rud [Rud, Theorem 9.21].

Theorem 0.3. *Let $f : \Omega \to \mathbb{R}$ be a continuous function on an open set $\Omega \subset \mathbb{R}^n$. Then f is continuously differentiable in the sense of Definition* d:def1 *0.1 if and only if it is continuously* *differentiable in the sense of Definition 0.2. If either of these conditions (hence both) hold then*

$$\mathbf{D}f(\mathbf{x}) = \left(\frac{\partial f}{\partial x_1}(\mathbf{x}), \ldots, \frac{\partial f}{\partial x_n}(\mathbf{x})\right), \qquad \textit{for all } \mathbf{x} \in \Omega.$$

The notion of continuous differentiability also extends to $\mathbb{R}^m$ functions.

Definition 0.4. Let $\Omega \subset \mathbb{R}^n$. We say that a continuous function $\mathbf{f} : \Omega \to \mathbb{R}^m$ is continuously differentiable in Ω if each of its components $f_i : \Omega \to \mathbb{R}$ is continuously differentiable in Ω for $i = 1, \ldots, m$, where $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}))$ for all $\mathbf{x} \in \Omega$. We define the *Jacobian of* $\mathbf{f}$ *at the point* $\mathbf{x}$ as the $m \times n$-matrix

$$J(\mathbf{f}, \mathbf{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}) & \ldots & \frac{\partial f_1}{\partial x_n}(\mathbf{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{x}) & \ldots & \frac{\partial f_m}{\partial x_n}(\mathbf{x}) \end{bmatrix}.$$

In other words, the (i, j)-th element of $J(\mathbf{f}, \mathbf{x})$ is $\frac{\partial f_i}{\partial x_j}(\mathbf{x})$.

We recall the inverse function theorem [Rud, Theorem 9.24].

Theorem 0.5 (Inverse function theorem)**.** *Let Ω be an open set in $\mathbb{R}^n$. Suppose $\mathbf{f} : \Omega \to \mathbb{R}^n$ be continuously differentiable. Let the $n \times n$-matrix $J(\mathbf{f}, \mathbf{a})$ be invertible for some $\mathbf{a} \in \Omega$ and $\mathbf{b} = \mathbf{f}(\mathbf{a})$. Then*

(a) there exist open sets U and V in $\mathbb{R}^n$ such that $\mathbf{a} \in U, \mathbf{b} \in V$ such that $\mathbf{f}$ is one-to-one on U and $\mathbf{f}(U) = V$.

(b) if $\mathbf{g} : V \to U$ be the inverse of $\mathbf{f}$ defined in V by (which exists by (a))

$$\mathbf{g}(\mathbf{f}(\mathbf{x})) = \mathbf{x}, \qquad \textit{for all } \mathbf{x} \in U,$$

then $\mathbf{g}$ is continuously differentiable in V. Furthermore

$$J(\mathbf{g}, \mathbf{y}) = J(\mathbf{f}, \mathbf{g}(\mathbf{y}))^{-1}, \qquad \textit{for all } \mathbf{y} \in V.$$

In particular, the matrix $J(\mathbf{g}, \mathbf{b})$ is the inverse of $J(\mathbf{f}, \mathbf{a})$, that is

$$J(\mathbf{g}, \mathbf{b}) = J(\mathbf{f}, \mathbf{a})^{-1}.$$

References

[Rud] W. Rudin. Principles of Mathematical Analysis, 3rd Edition.