1. Let $X$ be a reflexive normed vector space and let $T \in X^*$. Show that there exists $x_0 \in X$ such that $\|x_0\| = 1$ and

$$\|T\| := \sup_{x \in X, \|x\| = 1} \|Tx\| = Tx_0.$$ 

2. Let $X$ denote the space of continuous functions on $[0, 1]$ equipped with sup norm. Recall that $X$ is a Banach space.

(a) Show that 

$$Tf = \int_0^{1/2} f(x) \, dx - \int_{1/2}^1 f(x) \, dx$$ 

is a bounded linear functional.

(b) Compute $\|T\|$.

(c) Does there exist $f_0 \in X$ such that $\|f_0\| = 1$ and $\|T\| = Tf_0$?

(d) Is $X$ reflexive?

3. Let $X$ be a normed vector space and let $X^*$ denote the dual space. For a subspace $M$ of $X$, we set 

$$M^\perp := \{ f \in X^* : f(x) = 0 \quad \forall x \in M \}.$$ 

For a subspace $N$ of $X^*$, we define 

$$N^\perp := \{ x \in X : f(x) = 0 \quad \forall f \in N \}.$$ 

Prove that $(M^\perp)^\perp = \overline{M}$.


5. Consider the set 

$$C = \left\{ f \in L^1([0, 1]) : \int_0^1 f(t) \, dt = 1 \right\}.$$ 

Show that $C$ is a closed convex subset of $L^1([0, 1])$ which contains infinitely many elements of minimal norm. (Compare it to the previous question).