MATH 419/545
HW 6

Due April 8, 2020 (to be submitted online on Canvas by 11 AM)

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1. Assume p is aperiodic, irreducible and positive recurrent chain. Let π denote its unique stationary distribution. Prove that for any $x \in S$,

$$\lim_{n\to\infty} \sup_{A \in \mathcal{S}^{\mathbb{Z}_+}} |P_x((X_n, X_{n+1}, \ldots) \in A) - P_\pi(X \in A)| = 0.$$

2. Let p be an irreducible, aperiodic, null recurrent transition function on a countable set S. Prove that

$$\lim_{n\to\infty} p^n(i,j) = 0 \tag{1}$$

for all $i, j \in S$.
One approach is outlined below:
Let $Z_n = (X_n, Y_n)$ where X_n and Y_n are independent copies of the Markov chain with transition function p starting at an appropriate initial point. As we saw in the class Z is an S^2 valued Markov chain with transition function $\overrightarrow{p}((x_1, y_1), (x_2, y_2)) = p(x_1, x_2)p(y_1, y_2)$. We showed in class that $\overline{p}$ is irreducible as well.

   (a) If Z is transient, prove (1).

   (b) Assume Z is recurrent and (1) fails for some $i, j \in S$.

      (i) Show that there is a sequence $n_m \to \infty$, $\alpha_k \in [0,1]$ for all $k \in S$, with $\alpha_j > 0$, so that for any $i' \in S$, $\alpha_k = \lim_{m\to\infty} p^{n_m}(i', k)$. (The coupling theorem may help here).

      (ii) Verify α is a finite stationary measure for p and hence derive a contradiction. Hint: First use Fatou's lemma.

It is not hard to extend this result to the general setting where we de not assume that p is irreducible or aperiodic. Then for j null recurrent and any $i \in S$, $\lim_{n\to\infty} p^n(i,j) = 0$. But you do not have to prove this.

3. If B is a Brownian motion and $a > 0$, show that the process

$$\left\{ \exp\left(aB(t) - \frac{a^2 t}{2} \right) : t \geq 0 \right\}$$

is an $(\mathcal{F}_t^B)$-martingale.

4. The following exercise introduces continuous time ($\mathbb{R}$-valued, time homogeneous) Markov processes.
A function $p : [0, \infty) \times \mathbb{R} \times \mathcal{B} \to \mathbb{R}$, where $\mathcal{B}$ is the Borel σ-field on $\mathbb{R}$ is a **Markov transition kernel** provided

   (a) $p(\cdot, \cdot, A)$ is measurable as a function of (t, x) for each $A \in \mathcal{B}$.

   (b) $p(t, x, \cdot)$ is a Borel probability measure on $\mathbb{R}$ for each $t \geq 0$ and $x \in \mathbb{R}$.

   (c) (Chapman-Kolmorogov equation) for all $A \in \mathcal{B}, x \in \mathbb{R}$, $t, s > 0$,

$$p(t + s, x, A) = \int_{\mathbb{R}} p(t, y, A)\, p(s, x, dy).$$

Let $(\mathcal{F}_t)_{t \geq 0}$ be a filtration. An $(\mathcal{F}_t)$-adaoted process $\{X_t : t \geq 0\}$ is a $(\mathcal{F}_t)$-**Markov process** with transition kernel p, if for all $t \geq s$ and $A \in \mathcal{B}$, we have, almost surely,

$$P\left[X_t \in A | \mathcal{F}_s\right] = p(t - s, X(s), A).$$

Notice the similarity to the definition of (discrete time) Markov chain.

   (i) Let $\{B(t) : t \geq 0\}$ be a Brownian motion. Show that B is $(\mathcal{F}_t^B)$-Markov process (as defined above) and identify the Markov transition kernel.

   (ii) Let $\{B(t) : t \geq 0\}$ be a Brownian motion and set $X(t) = e^{-t} B(e^{2t})$ for all $t \geq 0$. Show that $\{X(t) : t \geq 0\}$ is a $(\mathcal{F}_t^X)$-Markov process (as defined above) and identify the Markov transition kernel.

Practice Problems (do not hand in)

1. Read the proofs of Lemma 5.6.4 and Lemma 5.6.5.
2. Exercise 5.6.1, 5.6.2, 5.6.3, 5.5.1
3. Exercise 7.1.2, 7.1.3, 7.1.6