MATH 419/545
HW 4

Due March 6, 2020 (at the beginning of the class)

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1. Let $(S, \mathcal{S}, P)$ be a probability space. Let $(S_i, \mathcal{S}_i)$ be measurable spaces for $i = 1, 2$. Let $X_i : (S, \mathcal{S}) \to (S_i, \mathcal{S}_i)$ be S_i-valued random variables for $i = 1, 2$. Let $\Phi : S_1 \times S_2 \to \mathbb{R}$ be a bounded measurable function with respect to the product σ-field $\mathcal{S}_1 \times \mathcal{S}_2$.

   (a) Show that for each $x \in S_1$, the function $\omega \to \Phi(x, X_2(\omega))$ is a measurable $\mathbb{R}$-valued function on $(S, \mathcal{S})$.

   (b) By part (a), for all $x \in S_1$, we can define

   $$H(x) := \int_S \Phi(x, X_2(\omega))\, dP(\omega).$$

   Show that the function $x \mapsto H(x)$ is a measurable $\mathbb{R}$-valued function on $(S_1, \mathcal{S}_1)$.

   (c) Let $\mathcal{G}$ be a sub-σ-field of $\mathcal{S}$ such that X_1 is $\mathcal{G}$-measurable and X_2 is independent of $\mathcal{G}$. Then show that

   $$E(\Phi(X_1, X_2)|\mathcal{G})(\omega) = H(X_1(\omega))$$

   P-almost surely.

   (Hint: Functional Montone Class Theorem for all the three parts)

2. (a) Let $(S, \mathcal{S})$ denote $(\mathbb{Z}_+, \mathcal{P}(Z_+))$ and let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $\{Y_{n,k} : (\Omega, \mathcal{F}) \to (S, \mathcal{S}) : n \in \mathbb{Z}_+, k \in \mathbb{N}\}$ be a collection of iid copies of $\mathbb{Z}_+$-valued random variables, where $(S, \mathcal{S}) = (\mathbb{Z}_+, \mathcal{P}(\mathbb{Z}_+))$. For $i \in S, A \in \mathcal{S}$ define $p : S \times \mathcal{S} \to [0, 1]$ by

   $$p(i, A) = P(\sum_{k=1}^{i} Y_{1,k} \in A).$$

   Show that p is a transtion probability on $(S, \mathcal{S})$

(b) Fix $x_0 \in \mathbb{Z}_+$ and define random variables $X_0 \equiv x_0$ and

$$X_{n+1} = \sum_{k=1}^{X_n} Y_{n,k}, \text{ for } n \in \mathbb{Z}_+.$$

Let $\mathcal{F}_n = \sigma\{Y_{k,j} : k < n, j \in \mathbb{N}\}$. Show that $\{X_n : n \in \mathbb{Z}_+\}$ is a Markov chain with respect to $(\mathcal{F}_n)$ with transtion probability p and initial distribution δ_{x_0}.

3. Consider a Markov chain $\{X_n : n \in \mathbb{Z}_+\}$ with general state space $(S, \mathcal{S})$ (not necessarily countable) defined on the cannonical product space $(\Omega, \mathcal{F}) = (S^{\mathbb{Z}_+}, \mathcal{S}^{\mathbb{Z}_+})$. If f is a bounded measurable function on S, let

$$Gf(x) = E_x(f(X_1)) - f(x).$$

G is called the **generator** of the Markov chain $\{X_n : n \in \mathbb{Z}_+\}$.

We say that h is a **harmonic function** on the set $D \in \mathcal{S}$ iff $Gh(x) = 0$ for all $x \in D$.

(a) If h is a bounded harmonic function on S prove that $h(X_n)$ is an $(\mathcal{F}_n^X)$-martingale with respect to every P_x. If $A \in \mathcal{S}$ and h is a bounded function on S which is harmonic on A^c, show that $h(X_{n \wedge V_A})$ is an $(\mathcal{F}_n^X)$-martingale with respect to every P_x. Here $V_A := \inf\{n \geq 0 : X_n \in A\}$ is as defined in Exercise 5.2.11

(b) Let $A \in \mathcal{S}$ satisfy $P_x(V_A < \infty) = 1$ for every $x \in S$. Let $f : A \to \mathbb{R}$ be bounded and measurable. Prove that $h(x) = E_x(f(X_{V_A}))$ is the unique bounded function on S which is harmonic on A^c and equals f on A.

(c) If $f : S \to \mathbb{R}$ is bounded and measurable, prove that $M_n^f = f(X_n) - \sum_{i=0}^{n-1} Gf(X_i)$ is an $(\mathcal{F}_n^X)$-martingale with respect to every P_x.

4. Exercise 5.2.6. (S is countable. You may assume that $k \in \mathbb{N}$).

5. Exercise 5.2.11. (S is countable. For (i) you need to verify that $g(x)$ is finite for all x. For (ii) you should verify the integrability of the martingale. For (ii) assume that the series in (*) is absolutely summable. For (iii) $g(x) = E_x\tau_A$ should read $g(x) = E_x V_A$ instead)

Practice Problems (do not hand in)

1. Let $Z : (\Omega, \mathcal{F}) \to (S, \mathcal{S})$ be a measurable map and let P be a probability measure on $(\Omega, \mathcal{F})$. Show that

   $$\nu(A) = P(Z^{-1}(A)) = P(\{Z \in A\}) = P(\{\omega \in \Omega : Z(\omega) \in A\}), \quad \text{for all } A \in \mathcal{S},$$

   defines a probability measure on $(S, \mathcal{S})$. This measure ν is called the 'pushforward measure of P under f'.

2. Let $p : S \to \mathcal{S} \to [0, 1]$ be a transition probabilty. Show that for all $f \in b\mathcal{S}$, the map

   $$x \mapsto \int_S f(y)\, p(x, dy)$$

   is also in $b\mathcal{S}$. (Hint: FMCT).

3. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $Y : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B})$ be a random variable. For all $x \in \mathbb{R}$, defined $p : \mathbb{R} \times \mathcal{B} \to [0, 1]$ be defined by

   $$p(x, A) = P(x + Y \in A).$$

   Let $x_0 \in \mathbb{R}$. Let $Y_j : \Omega \to \mathbb{R}$ be iid copies of Y, for $j \in \mathbb{N}$ and let $\mathcal{F}_n = \mathcal{F}_n^Y = \sigma(Y_1, \ldots, Y_n)$. Let $X_n = x_0 + \sum_{j=1}^n Y_j$. Show that p is a transition probability on $(\mathbb{R}, \mathcal{B})$ and that X_n is a Markov chain with respect to $\mathcal{F}_n$ with transition probability p.

4. Let $\{X_n : n \in \mathbb{Z}_+\}$ be a Markov chain on $(\Omega, \mathcal{F}, P)$ with respect to the filtration $(\mathcal{F}_n)$. Show that the map $X : (\Omega, \mathcal{F}) \to (S^{\mathbb{Z}_+}, \mathcal{S}^{\mathbb{Z}_+})$ defined by

   $$\omega \mapsto (X_0(\omega), X_1(\omega), \ldots)$$

   is measurable. (Hint: $\pi - \lambda$ theorem).
   Remark: Practice problems 1. and 4. can be used to justify the probability measure on the cannonical space (the measure P_X on $(S^{\mathbb{Z}_+}, \mathcal{S}^{\mathbb{Z}_+})$ given the initial distribution μ and the transition probability p defined in class is unique). The measure P_X defined in class is simply the pushforward of P under the map X.

5. Exercise 5.2.7.

6. Let $\{X_n : n \in \mathbb{Z}_+\}$ be a stochastic process and let $(\mathcal{F}_n)_{n\in\mathbb{Z}_+}$ be a filtration on $(\Omega, \mathcal{F}, P)$. Let $S, T : \Omega \to \mathbb{Z}_+ \cup \{\infty\}$ be $\mathcal{F}_n$-stopping times. Show the following:

   (a) If X_∞ is $\mathcal{F}$-measurable and X_n is $(\mathcal{F}_n)$-adapted, then $\omega \mapsto X_{T(\omega)}(\omega)$ is $\mathcal{F}_T$-measurable.

   (b) If $S \le T$ then $\mathcal{F}_S \subseteq \mathcal{F}_T$.

   (c) $\mathcal{F}_{S\wedge T} = \mathcal{F}_S \cap \mathcal{F}_T$.

   (d) If $F \in \mathcal{F}_{S\vee T}$ then $F \cap \{S \le T\} \in \mathcal{F}_T$.

   (e) $\mathcal{F}_{S\vee T} = \sigma(\mathcal{F}_S \cup \mathcal{F}_T)$.

7. Let $X_i : (\Omega, \mathcal{F}) \to (S_i, \mathcal{S}_i)$ be measurable for $i = 1, 2$. Show that the map $\omega \mapsto (X_1(\omega), X_2(\omega))$ is a measurable map from $(\Omega, \mathcal{F})$ to the product space with product σ-field $(S_1 \times S_2, \mathcal{S}_1 \times \mathcal{S}_2)$ (we used this fact in the proof of Strong Markov property. Hint: π-λ theorem).

8. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{P}$ be a π-system such that $\mathcal{P} \subset \mathcal{F}, \Omega \in \mathcal{F}$. Let $\mathcal{G} = \sigma(\mathcal{P})$. Let X and Y be two integrable random variables such that X is $\mathcal{F}$-measurable and Y is $\mathcal{G}$-measurable. Suppose that

$$\int_A X\,dP = \int_A Y\,dP \quad \text{for all } A \in \mathcal{P},$$

then show that

$$E(X|\mathcal{G}) = Y \quad \text{almost surely.}$$