MATH 419/545
HW 2

Due January 31, 2019 (at the beginning of the class)

Notation: $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}, \mathbb{N} = \{1, 2, 3, \ldots\}$.

1. Assume $\{X_n : n \in \mathbb{Z}_+\}$ are independent $\{0,1\}$-valued random variables such that $P(X_n = 1) = 2^{-n}$ for all $n \in \mathbb{Z}_+$. Define

$$L = \sup \{n \in \mathbb{Z}_+ : X_n = 1\}.$$

   (a) Prove that $L < \infty$ almost surely.

   (b) Prove that L is not a $(\mathcal{F}_n^X)$-stopping time.

2. Let $\{Y_i : i \in \mathbb{N}\}$ be independent mean 0 random variables such that $|Y_i| \leq K$ for all $i \in \mathbb{N}$. Let $S_n = \sum_i^n Y_i$. Prove that either
   (a) S_n converges almost surely or
   (b) $P(\{\limsup_{n\to\infty} S_n = \infty \text{ and } \liminf_{n\to\infty} S_n = -\infty\}) = 1$.

3. Exercise 4.2.5

4. Exercise 4.2.6 (for part (ii) assume that $E|\log Y_1| < \infty$).

5. Exercise 4.6.4

Practice Problems (do not hand in)

1. Try to prove (or learn the proofs from the text) Theorem 4.1.10 (Conditional Jensen inequality), and Theorem 4.6.2 (a sufficient condition for uniform integrability).

2. Read the proof of Theorem 4.2.11 (Martingale convergence theorem).

3. TRUE OR FALSE: If $(X_n : n \in \mathbb{N})$ is a $(\mathcal{F}_n)$-submartingale, then so is $(X_n^+ : n \in \mathbb{N})$.

4. If S, T are $(\mathcal{F}_n)$-stopping times then show that $S \vee T, S \wedge T$ and $S + T$ are also $(\mathcal{F}_n)$-stopping times.

5. Exercises 4.2.1, 4.2.2, 4.2.3, 4.2.4.

6. Let $N : \Omega \to \mathbb{N}$ denote a random variable with $P(N = n) = \frac{c}{n^2}$, where $c > 0$ is chosen so that $\sum_{n\in\mathbb{N}} cn^{-2} = 1$. Let $X_i : i \in \mathbb{N}$ be iid random variables that are independent of N such that $P(X_i = 1) = P(X_i = -1) = \frac{1}{2}$. Let $\mathcal{F}_n, n \in \mathbb{N}$ denote the σ-field $\sigma(N, X_1, \ldots, X_n)$. Let $S_n = \sum_{k=1}^{n} X_k, T = \inf\{n \in \mathbb{N} : |S_n| = N\}$. Show that

   (a) S_n is an $\mathcal{F}_n$-martingale.

   (b) T is a $\mathcal{F}_n$-stopping time.

   (c) The martingale $S_{n\wedge T}$ converges almost surely, but $E(|S_{n\wedge T}|) = EN = \infty$. (This shows that unlike Theorem 4.2.11, one cannot take $E|X| < \infty$ in Theorem 4.3.1 even if $P(C) = 1$).

7. Give an example of a collection of random variables that is L^1-bounded but not uniformly integrable.