MATH 419/545
HW 1

Due January 17 (at the beginning of the class)

The numbered exercises below are from the textbook: see
`https://services.math.duke.edu/~rtd/PTE/PTE5_011119.pdf`.

To be handed in

1. Let μ, ν be measures on $(\Omega, \mathcal{F})$ such that μ is probability measure, and ν is a finite measure. We say that ν is **uniformly absolutely continuous** with respect to μ, if for all $\epsilon > 0$ there exists $\delta > 0$ such that any $A \in \mathcal{F}$ with $\mu(A) < \delta$ satisfies $\nu(A) < \epsilon$.

   Show that ν is uniformly absolutely continuous with respect to μ **if and only if** ν is absolutely continuous with respect to μ.

2. Exercise 4.1.5
3. Exercise 4.1.6
4. Exercise 4.1.7
5. Exercise 4.1.8
6. Exercise 4.1.9

Practice problems (do not hand in)

1. Exercises 4.1.2, 4.1.3, 4.1.4.
2. Let $(\Omega, \mathcal{F})$ be a measurable space and let $X, Y : \Omega \to \mathbb{R}$ be random variables. Show that the following are equivalent:

   (a) Y is $\sigma(X)$ measurable;

   (b) There exists a Borel function $h : \mathbb{R} \to \mathbb{R}$ such that $Y = h(X)$.

   (This is called the **Doob-Dynkin lemma**)

3. Let X be an integrable random variable on $(\Omega, \mathcal{F})$. Let $\mathcal{G}$ denote the σ-field $\{\emptyset, \Omega\}$. Then show that the conditional expectation generalizes the notion expectation in the following sense:

   $$\mathbb{E}(X|\mathcal{G})(\omega) = \mathbb{E}X = \int_\Omega X \, dP,$$

   for all $\omega \in \Omega$, where $\mathbb{E}X$ is the expectation of X.

4. The purpose of this exercise is to understand some formulae for conditional probability in basic probability courses as special cases of our (measure theoritic) definition of conditional probability.

- Let B be an event in $(\Omega, \mathcal{F}, P)$ with $0 < P(B) < 1$. Prove that

$$\mathbb{E}[1_A|\sigma(1_B)](\omega) = \frac{P(A \cap B)}{P(B)} 1_B(\omega) + \frac{P(A \cap B^c)}{P(B^c)} 1_{B^c}(\omega), \quad \text{for all } \omega \in \Omega.$$

  Remark: This explains the reason for the definition $P(A|B) = \frac{P(A \cap B)}{P(B)}$ and Bayes' rule in basic probability courses.

- Let X, Y be random variables with joint probability density function (pdf) $f_{X,Y} : \mathbb{R} \times \mathbb{R} \to [0, \infty)$, that is

$$P(X \leq x, Y \leq y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(s,t)\, ds\, dt, \quad \text{for all } x \in \mathbb{R}, y \in \mathbb{R}.$$

  Define the *conditional probability density function*

$$f_{X|Y=y}(x) = \begin{cases} \frac{f_{X,Y}(x,y)}{\int_{\mathbb{R}} f_{X,Y}(s,y)\, ds} & \text{if } \int_{\mathbb{R}} f_{X,Y}(s,y)\, ds \neq 0, \\ 0 & \text{if } \int_{\mathbb{R}} f_{X,Y}(s,y)\, ds = 0,. \end{cases}$$

  For all $y \in \mathbb{R}$, show that the function $f_{X|Y=y} : \mathbb{R} \to \mathbb{R}$ is Borel measurable. For any bounded Borel measurable function $g : \mathbb{R} \to \mathbb{R}$, show that

$$\mathbb{E}[g(X)|\sigma(Y)] = h(Y),$$

  where $h : \mathbb{R} \to \mathbb{R}$ is the function

$$h(y) = \int_{\mathbb{R}} f_{X|Y=y}(x) g(x)\, dx.$$

  Show that $\mathbb{E}[g(X)|\sigma(Y)] = h(Y)$ holds even if we assume g to be a Borel measurable function (not necessarily bounded) such that $g(X)$ is integrable. (You should compare with Doob-Dynkin lemma)

Instructions on submitting Homework:

1. Solutions will be graded both on accuracy and quality of exposition. Solutions should be mathematically rigorous, well-crafted, and written in complete English sentences. Solutions must always be legible; use of LaTeX is encouraged and appreciated.

2. Please staple your pages together when you submit your assignment.