My research interests lie at the interface of probability and analysis. The broad goal of my research is to understand how the long-term behavior of Markov chains depends on the large-scale geometry of the underlying state space. In particular, I am interested in heat kernel estimates for discrete time Markov chains on state spaces having a geometric structure, for instance a weighted graph, Riemannian manifold or more generally a metric measure space.

Functional inequalities (for example Sobolev, Nash, Poincaré and isoperimetric inequalities) and regularity theory of PDE (Harnack inequalities, Hölder regularity) are major themes in my work. An important motivation for studying transition probability estimates and their relationship with functional inequalities is due to their robustness. For instance, these functional inequalities are preserved under quasi-isometries of the state space and small perturbations of the Markov kernel. In particular, the arguments we develop are robust with respect to perturbations of the geometry of the underlying space; for example addition or removal of few edges in a graph or small changes in the metric of a Riemannian manifold.

This research statement is divided into three sections and describes my work in the following areas: Gaussian estimates for random walks, anomalous threshold behavior of heavy-tailed random walks and stabilizability of divisible sandpiles. I will highlight some of my results in the above areas. At the end of each section, I will describe some related problems and specific conjectures that I plan to investigate. Each section is self-contained and may be read independently of others.

1 Gaussian estimates for random walks

It is well known that two sided Gaussian bounds on heat kernel can be characterized by geometric properties. In particular, the following equivalence is known under various settings like diffusion on Riemannian manifolds [Gri91, Sal92], diffusion on local Dirichlet spaces [Stu96] and random walks on graphs [Del99]. The following are equivalent:

1. Two sided Gaussian bounds on heat kernel
2. A scale invariant Parabolic Harnack inequality
3. Volume doubling property and a scale invariant Poincaré inequality.

The hardest and the most useful implication is (3) implies (1) and (2). The conditions on volume growth and Poincaré inequalities are easy to verify given the geometric data of the space. What is particularly useful about this equivalence is the fact that it is easier to show stability of (3) under quasi-isometries than to show such stability properties directly for (1) and (2). In a joint work [MS1] with Saloff-Coste, we prove the above equivalence for random walks on a large class of state spaces.

One of our goals was to find a unified framework that includes both continuous spaces like Riemannian manifolds and discrete spaces like graphs. Quasi-geodesic spaces provide us with such a setting. Quasi-geodesic spaces are spaces quasi-isometric to graphs. They have desired ‘connectivity properties’ and are broad enough to cover both continuous and discrete spaces.

To precisely describe the results, let us recall some basic notions of discrete time Markov chains on a metric measure space \((M, d, \mu)\). We denote the closed balls by \(B(x, r) := \{y \in M : d(x, y) \leq r\}\) and the volume of balls by \(V_\mu(x, r) := \mu(B(x, r))\). We consider symmetric Markov chains \((X_k)_{k \in \mathbb{N}}\) whose one-step transition kernel \(p_1(x, y)\) with respect to
the measure $\mu$ satisfies the following properties: $p_1$ is $\mu$-symmetric, that is $p_1(x,y) = p_1(y,x)$ for all $x,y \in M$. Moreover, we assume that there exists $C_0 > 0$ such that
\[ C_0^{-1} \frac{1_{B(x,1)}(y)}{V_\mu(x,1)} \leq p_1(x,y) \leq C_0 \frac{1_{B(x,1)}(y)}{V_\mu(x,1)} \tag{1.1} \]
for all $x, y \in M$. The above assumption can be weakened in several cases and is only a simple illustrative example. We are interested in the following two-sided Gaussian estimates that
\[ p_{x,y} \quad \forall x,y \in M. \]

The main result of [MS1] is a characterization of the above two-sided Gaussian estimates. A statement of the characterization skipping technical details is given below.

**Theorem 1.1.** Let $(M, d, \mu)$ be quasi-geodesic metric measure space $(M, d, \mu)$ and suppose that $p_1$ is a symmetric transition density with respect to $\mu$ satisfying (1.1). Then conditions (1), (2), (3) are equivalent.

I will describe some concrete motivating examples to which the equivalence applies and how it compares to the previously known results. Apart from generalizing Delmotte’s theorem on graphs, there are several new examples for which we obtain two sided Gaussian estimates. For instance, natural random walks (see ball walk of Example 1.4) on a Riemannian manifold with positive injectivity radius and non-negative Ricci curvature satisfy two sided Gaussian estimates. A similar result was previously known for diffusion. However, there are some subtle differences between Gaussian estimates for diffusions and random walks. The examples below illustrate this and other subtleties involved. Most of these examples are for state spaces as simple as $\mathbb{R}^n$ for which two sided Gaussian estimates were previously unknown!

**Example 1.2.** Let $(M, d, \mu)$ be a metric measure space. We define $q_1(x,y) = \frac{1_{B(x,1)}(y)}{\sqrt{V_\mu(x,1)V_\mu(y,1)}}$. Note that $q_1$ need not be a Markov kernel since $\int_M q_1(x,y)\mu(dy)$ is not necessarily 1. However under mild assumptions on $(M, d, \mu)$, there exists $C_3 > 0$ such that
\[ \sigma(x) := \left( \int_M q_1(x,y)\mu(dy) \right)^{-1} \in (C_3^{-1}, C_3) \]
for all $x \in M$. Hence the kernel $p_1(x,y) := q_1(x,y)/(\sigma(x)\sigma(y))$ is a symmetric Markov kernel with respect to the measure $\hat{\mu} \ll \mu$ defined by $d\hat{\mu}/d\mu(x) = \sigma(x)$. Since $\sigma$ is bounded between two positive reals, $(M, d, \hat{\mu})$ satisfies volume doubling and Poincaré inequality if and only if $(M, d, \mu)$ satisfies volume doubling and Poincaré inequality. Thus by the main result of [MS1], $p_k$ satisfies two sided Gaussian estimates if and only if $(M, d, \mu)$ satisfies volume doubling and Poincaré inequality.
Example 1.3. We expand upon Example 1.2 in specific instances. Let $d_E$ denote the Euclidean distance in $\mathbb{R}^n$ and let $\lambda_n$ denote the Lebesgue measure in $\mathbb{R}^n$. By the main results of [HS93], we have Gaussian estimates for the kernel $p_n$ in Example 1.2 if $(M, d, \mu) = (\mathbb{R}^n, d_E, \lambda_n)$. However if we replace the measure $\lambda_n$ by a slightly perturbed measure $f \, d\lambda_n$ where $f$ is a non-constant measurable function on $\mathbb{R}^n$ bounded between two positive constants, then two sided Gaussian bounds were previously unknown. In particular, the proof of Gaussian lower bounds in [HS93] does not work. However our methods in [MS1] being robust with respect to these perturbations, allows us to obtain Gaussian estimates for $p_k$.

For $\alpha \in \mathbb{R}$, we define a measure $\lambda_{\alpha,n}$ on $\mathbb{R}^n$ such that $\lambda_{\alpha,n} \ll \lambda_n$ and $\frac{d\lambda_{\alpha,n}}{d\lambda_n}(x) := (1 + \|x\|_2^2)^{\alpha/2}$, where $\|x\|_2$ denote the Euclidean norm of $x$. Canonical diffusion on $(\mathbb{R}^n, d_E, \lambda_{\alpha,n})$ was first studied in [GS05]. We develop comparison techniques in [MS1] to transfer Poincaré inequalities in the diffusion setting to Poincaré inequalities in the random walk setting and vice-versa. Using these comparison techniques along with results of [GS05], we know the values of $\alpha$ and $n$ for which $p_k$ in Example 1.1 has Gaussian estimates on $(\mathbb{R}^n, d_E, \lambda_{\alpha,n})$. Some examples and non-examples of volume doubling and Poincaré inequality are summarized in the table below:

<table>
<thead>
<tr>
<th>Volume doubling</th>
<th>Poincaré inequality</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>True</td>
<td>True</td>
<td>$(\mathbb{R}^n, d_E, \lambda_{\alpha,n})$ with $n \geq 2$ and $\alpha &gt; -n$ or $n = 1$ and $\alpha \in (-1, 1)$</td>
</tr>
<tr>
<td>True</td>
<td>False</td>
<td>$(\mathbb{R}^n, d_E, \lambda_{\alpha,1})$ with $\alpha \geq 1$</td>
</tr>
<tr>
<td>False</td>
<td>True</td>
<td>$(\mathbb{R}^n, d_E, \lambda_{\alpha,n})$ with $\alpha \leq -n$</td>
</tr>
<tr>
<td>False</td>
<td>False</td>
<td>Hyperbolic space, 3-regular tree</td>
</tr>
</tbody>
</table>

Example 1.4. A natural random walk on $(M, d, \mu)$ is given by the Markov kernel $q_1(x, y) = 1_{B(x,1)}(y)/\mu(x, 1)$. Although $q_1$ is a Markov kernel it is not necessarily symmetric with respect to $\mu$ since $q_1(x, y) \neq q_1(y, x)$ in general. However the random walk is symmetric with respect to the measure $\mu' \ll \mu$ defined by $\frac{d\mu'}{d\mu}(x) = V_\mu(x, 1)$; its Markov kernel with respect to $\mu'$ is $p_1(x, y) = \frac{1_{B(x,1)}(y)}{V_{\mu'}(x, 1)\mu'(x, 1)}$. We call this chain the ball walk on $(M, d, \mu)$. Such random walks on compact Riemannian manifolds were studied in [LM10].

What is perhaps surprising is that the measures $\mu'$ and $\mu$ are not comparable in general and this might lead to different long-term behaviors for canonical diffusion and the ball walk. We illustrate this for $(\mathbb{R}^3, d_E, \lambda_{-2,3})$. By the results of [GS05], the transition probability density with respect to $\lambda_{-2,3}$ for the canonical diffusion satisfies two sided Gaussian bounds. However the ball walk on $(\mathbb{R}^3, d_E, \lambda_{-2,3})$ is symmetric with respect to a measure comparable to $\lambda_{-4,3}$. By the table in Example 1.3, the kernel of the ball walk on $(\mathbb{R}^3, d_E, \lambda_{-2,3})$ with respect to its symmetric measure does not satisfy Gaussian bounds.

I would like to highlight that Theorem 1.1 is not an easy generalization of Delmotte’s work on graphs [Del99]. This is because discrete time Markov chains on continuous spaces introduces new difficulties that are not present in previous settings. I will illustrate one of these difficulties. The works of Saloff-Coste [Sal92], Delmotte [Del99] and Sturm [Stu96] rely on the Moser’s iteration method to prove a parabolic Harnack inequality (see [Mos61, Mos64] for Moser’s work). Along with Poincaré inequality and volume doubling, Moser’s iteration relies on repeated applications of a Sobolev inequality. The Sobolev inequalities in the previous settings are of the form

$$\|f\|_2^{2\delta/(\delta - 2)} \leq \frac{Cr^2}{V_\mu(x, r)^{2/\delta}} \left(\mathcal{E}(f, f) + r^{-2} \|f\|_2^2\right)$$

(1.2)
for all ‘nice’ functions $f$ supported in $B(x,r)$. However for discrete time Markov chains, the Dirichlet form satisfies the inequality $\mathcal{E}(f,f) = \langle (I - P)f, f \rangle \leq 2 \|f\|_2^2$. However this means that $L^2(B(x,r)) \subset L^{2^{\delta - 2}}(B(x,r))$ for all balls $B(x,r)$ which can happen only if the space is discrete. Hence for discrete time Markov chains on continuous spaces the Sobolev inequality (1.2) cannot possibly be true. We prove and rely on a weaker form of the Sobolev inequality (1.2) which seems to be too weak to run Moser’s iteration directly to prove parabolic Harnack inequality. Instead we use Moser’s iteration to prove a version of the mean value inequality which in turn gives Gaussian upper bounds. The proof of Gaussian lower bounds requires methods that are similar to [HS02] which uses elliptic Harnack inequality and Gaussian upper bounds to prove Gaussian lower bounds. Our work implicitly contains an alternate proof of Delmotte’s work on graphs.

In [MS1], various known applications of Gaussian estimates and parabolic Harnack inequalities are extended in greater generality. They include law of iterated logarithm, estimates on Green’s functions, Hölder continuity of harmonic and caloric functions, Liouville properties, estimates on mixing times and dimension of harmonic functions with polynomial volume growth. Moreover stability under quasi-isometries of Poincaré inequality and volume doubling in a general context was proved. This implies the stability of the parabolic Harnack inequality and two sided Gaussian estimates under quasi-isometry.

### 1.1 Future directions

One of the features of our work is that it provides an unified approach to Gaussian estimates for discrete time Markov chains on both discrete and continuous spaces. I would like to develop a similar theory to study sub-Gaussian estimates for discrete time Markov chains on a large class of metric measure spaces. There are analogous characterizations for sub-Gaussian estimates have been developed in the setting of diffusions on local Dirichlet spaces [GT12] and random walks on graphs [GT02, BCK05]. As before there are some difficulties with Sobolev inequalities for general setup, especially continuous spaces. Moreover, there are some new functional inequalities (like cut-off Sobolev inequalities) required to characterize sub-Gaussian estimates.

**Problem 1.5.** Extend characterizations of sub-Gaussian estimates for discrete time Markov chains for a large class of metric measure spaces (to include graphs and continuous spaces).

In [CM97], Colding and Minicozzi proved that the space of harmonic functions with polynomial volume growth with fixed rate on a manifold satisfying volume doubling and Poincaré inequality is finite dimensional. A recent surprising application of this result is an alternate proof of Gromov’s theorem on groups of polynomial volume growth due to Kleiner [Kle10]. This new proof avoids the solution to Hilbert’s fifth problem (Montgomery-Zippin-Yamabe structure theory). I would like to extend Colding and Minicozzi’s result for harmonic functions with respect to the Markov kernel (satisfying $Ph = h$ where $P$ is Markov operator) to metric measure spaces satisfying volume doubling and Poincaré inequality.

**Conjecture 1.** For quasi-geodesic metric measure spaces satisfying volume doubling and Poincaré inequality, the space of harmonic functions with polynomial volume growth of a fixed rate is finite dimensional.
2 Anomalous threshold behavior of long range random walks

In this section, I will describe two joint works with Saloff-Coste [MS2, MS3] on random walks with heavy-tailed jumps. In [MS3], we consider weighted graphs satisfying sub-Gaussian estimate for the natural random walk. This is typical of ‘pre-fractal’ graphs like Sierpinski gaskets, carpets and Viscek graphs. See [Kum14] for a recent survey on such anomalous random walks. On these graphs, we study symmetric Markov chains with heavy-tailed jumps. We establish a new threshold behavior of such Markov chains when the index governing the tail heaviness (or jump index) equals the escape time exponent (or walk dimension) of the sub-Gaussian estimate. In a certain sense, this generalizes the classical threshold corresponding to the second moment condition.

In much of the existing literature the ‘jump index’ $\beta$ is assumed to be in $(0, 2)$ (see for example [BL02, CK03a, CK03b, BBK09]). Our work is a modest attempt to understand the behavior of such random walks for all $\beta \in (0, \infty)$. Similar results are obtained using different methods in [SZ13] for random walks on groups and their corresponding second moment threshold behavior.

Let $\Gamma$ be vertices of an infinite, connected, locally finite graph endowed with a weight $\mu_{xy}$. The weight is a symmetric non-negative function on $\Gamma \times \Gamma$ such that $\mu_{xy} > 0$ if and only if $x$ and $y$ are neighbors (in which case we write $x \sim y$). We call the pair $(\Gamma, \mu)$ a weighted graph. The weight $\mu_{xy}$ on the edges induces a weight $\mu(x)$ on the vertices and a measure $\mu$ on subsets $A \subset \Gamma$ defined by

$$\mu(x) := \sum_{y, y \sim x} \mu_{xy} \quad \text{and} \quad \mu(A) := \sum_{x \in A} \mu(x).$$

Let $d(x, y)$ be the graph distance between points $x, y \in \Gamma$, that is the minimal number of edges in any edge path connecting $x$ and $y$. Denote the metric balls by $B(x, r) := \{y \in M : d(x, y) \leq r\}$ and their measures as follows $V(x, r) := \mu(B(x, r))$. for all $x \in \Gamma$ and $r \geq 0$. We consider weighted graphs $(\Gamma, \mu)$ satisfying the following uniform volume doubling assumption: there exists $V_h : [0, \infty) \to (0, \infty)$, a strictly increasing function and constants $C_D, C_h > 1$ such that

$$V_h(2r) \leq C_D V_h(r), \forall r > 0 \quad \text{and} \quad C_h^{-1} V_h(r) \leq V(x, r) \leq C_h V_h(r), \forall x \in M, \forall r > 0. \quad (2.1)$$

There is a natural random walk $X_n$ on $(\Gamma, \mu)$ associated with the edge weights $\mu_{xy}$. The Markov chain is defined by the one-step transition probability $P(x, y) = \mathbb{P}(X_1 = y) = \mu_{xy}/\mu(x)$. We will assume that there exists $p_0 > 0$ such that $P(x, y) \geq p_0$ for all $x, y$ such that $x \sim y$. For any non-negative integer $n$, the $n$-step transition probability $P_n$ is defined by $P_n(x, y) = \mathbb{P}(X_n = y \mid X_0 = x) = \mathbb{P}^x(X_n = y)$. Define the heat kernel of weighted graph $(\Gamma, \mu)$ by

$$p_n(x, y) := \frac{P_n(x, y)}{\mu(y)}.$$

We assume that there exists $\gamma > 1$ such that the following sub-Gaussian estimates are true for the heat kernel $p_n$. There exists constants $c, C > 0$ such that, for all $x, y \in \Gamma$

$$p_n(x, y) \leq \frac{C}{V_h(n^{1/\gamma})} \exp \left[ - \left( \frac{d(x, y)^\gamma}{cn} \right)^{1/\gamma} \right], \forall n \geq 1 \quad (2.2)$$

and

$$(p_n + p_{n+1})(x, y) \geq \frac{c}{V_h(n^{1/\gamma})} \exp \left[ - \left( \frac{d(x, y)^\gamma}{cn} \right)^{1/\gamma} \right], \forall n \geq 1 \vee d(x, y). \quad (2.3)$$
The parameter $\gamma$ in (2.2) and (2.3) is sometimes called the ‘escape time exponent’ or ‘anomalous diffusion exponent’ or ‘walk dimension’. It is known that $\gamma \geq 2$ necessarily. For any $\alpha \in [1, \infty)$ and for any $\gamma \in [2, \alpha + 1]$, Barlow constructs graphs of polynomial volume growth satisfying $V(x,r) \simeq (1+r)^\alpha$ and sub-Gaussian estimates (2.2) and (2.3) (see Theorem 2 of [Bar04]). Moreover, these are the complete range of $\alpha$ and $\gamma$ for which sub-Gaussian estimates with escape rate exponent $\gamma$ could possibly hold for graphs of polynomial growth with growth exponent $\alpha$.

We demonstrate threshold behavior as the jump index $\beta$ varies by analyzing the function $n \mapsto \sup_{x \in \Gamma} k_{2n}(x,x) = \sup_{x,y \in \Gamma} k_{2n}(x,y)$ as $n$ goes to infinity, where $k_n$ denote the $n$-step transition probability density with respect to $\mu$. The main result of [MS3] is the following matching upper and lower estimates on $\sup_{x \in \Gamma} k_{2n}(x,x)$.

**Theorem 2.1.** [MS3] Let $(\Gamma, \mu)$ be a weighted graph satisfying the homogeneous volume doubling condition (2.1) and suppose that its heat kernel $p_n$ satisfies the sub-Gaussian bounds (2.2) and (2.3) with escape time exponent $\gamma$. Let $K$ be a Markov operator symmetric with respect to the measure $\mu$ whose one step transition probability density with respect to $\mu$ satisfies

$$\frac{C_1^{-1}}{V_h(d(x,y))\phi(d(x,y))} \leq k_1(x,y) = k_1(y,x) \leq \frac{C_1}{V_h(d(x,y))\phi(d(x,y))},$$

for all $x, y \in M$, where $\phi : [0, \infty) \to [1, \infty)$ is a continuous regularly varying function of positive index and $C_1 > 0$ is a constant. Then there exists a constant $C > 0$ such that

$$\frac{C^{-1}}{V_h(\zeta(n))} \leq \sup_{x \in \Gamma} k_{2n}(x,x) = \sup_{x,y \in \Gamma} k_{2n}(x,y) \leq \frac{C}{V_h(\zeta(n))},$$

for all $n \in \mathbb{N}$, where $\zeta : [0, \infty) \to [1, \infty)$ is a continuous non-decreasing function which is an asymptotic inverse of $t \mapsto t^{\gamma} / \int_0^t s^{\gamma-1} \frac{ds}{\phi(s)}$.

In particular, if $k_1(x,y) \simeq \frac{1}{V_h(d(x,y))(1+d(x,y))^{\gamma}}$, then by Theorem 2.1 we see the threshold behavior when $\beta = \gamma$ as

$$\sup_{x \in \Gamma} k_{2n}(x,x) \simeq \begin{cases} \frac{1}{V_h(n^{1/\beta})} & \text{if } \beta < \gamma, \\ \frac{1}{V_h((n \log(2+n))^{1/\gamma})} & \text{if } \beta = \gamma, \\ \frac{1}{V_h(n^{1/\gamma})} & \text{if } \beta > \gamma. \end{cases}$$

Here and elsewhere in this document, the symbol $\simeq$ means ‘bounded above and below by positive constants’. A corollary of Theorem 2.2 is that a heavy-tailed random walk has the same return probability behavior as simple random walk if and only if it has finite $\gamma$-moment.

In [MS2], we consider a uniformly discrete metric measure space $(M,d,\mu)$ satisfying space homogeneous volume doubling condition (2.1) and consider random walks with heavy-tailed jumps with jump index $\beta \in (0,2)$. Extending several existing works by other authors [BL02, CK03a, CK03b, BBK09], we prove the following global upper and lower bounds for $n$-step transition probability density that are sharp up to constants.

**Theorem 2.2.** [MS2] Let $(M,d,\mu)$ be a uniformly discrete metric measure space satisfying the space homogeneous volume doubling condition (2.1). Consider a random walk symmetric
with respect to measure $\mu$ whose one-step transition density $k_1$ with respect to $\mu$ satisfies

$$k_1(x, y) = k_1(y, x) \simeq \frac{1}{(V_h(d(x, y))((1 + d(x, y))^l(d(x, y))))^\beta}$$

where $l$ is a continuous, positive, slowly varying function with the jump index $\beta \in (0, 2)$ and $V_h$ is the homogeneous volume growth function. Then the $n$-step transition density $k_n$ with respect to $\mu$ satisfies

$$k_n(x, y) \simeq \left( \frac{1}{V_h(n^{1/\beta}l_#(n^{1/\beta}))} \wedge \frac{n}{V_h(d(x, y))\phi(d(x, y))} \right),$$

where $l_#$ is the de Bruijn conjugate of $l$.

2.1 Future directions

I plan to investigate these questions posed in [MS2, MS3]:

**Problem 2.3.** Generalize Theorem 2.2 if the homogeneous volume doubling assumption given by (2.1) is replaced by the more general volume doubling condition: there exists $C_D > 1$ such that $V(x, 2r) \leq C_D V(x, r)$ for all $x \in M$ and for all $r > 0$.

Although it seems like a minor technical issue, Problem 2.3 is challenging and its solution will require novel ideas. In particular, we need to develop local alternatives to global version of functional inequalities like the Nash inequality. In [MS3], we raise several conjectures and some of which are listed below. The first is to find matching upper and lower bounds for transition density for the case $\beta \in (0, \gamma)$.

**Conjecture 2.** Consider a weighted graph $(\Gamma, \mu)$ satisfying the homogeneous volume doubling condition and sub-Gaussian estimate with escape time exponent $\gamma$. Consider a random walk symmetric with respect to measure $\mu$ whose one-step transition density $k_1$ with respect to $\mu$ satisfies

$$k_1(x, y) = k_1(y, x) \simeq \frac{1}{(V_h(d(x, y))((1 + d(x, y))^l(d(x, y))))^\beta}$$

where $l$ is a continuous, positive, slowly varying function with the jump index $\beta \in (0, \gamma)$ and $V_h$ is the homogeneous volume growth function. Then the $n$-step transition density $k_n$ satisfies the estimate (2.5).

Note that Theorem 2.2 gives the desired estimate for $\beta \in (0, 2)$. The proof of (2.5) in [MS2] doesn’t seem to work if $\beta \in [2, \gamma)$. In particular, the use of Davies’ method to prove off-diagonal upper bounds do not work and one requires new techniques. In [MS3], we provide evidence towards the above two questions by an example constructed using stable subordinator. Another intriguing question is to find matching two-sided estimates $k_n(x, y)$ for the case $\beta \geq \gamma$ for appropriate range of $d(x, y)$. In light of [SZ13, Theorem 1.5] for random walks on groups, we conjecture:

**Conjecture 3** (Near diagonal estimate). Under the assumptions of Theorem 2.1 with the additional assumption $\beta > \gamma$, we have

$$k_n(x, y) \simeq \frac{1}{V_h(n^{1/\gamma})}$$

for all $n \in \mathbb{N}^*$ and for all $x, y \in M$ such that $d(x, y) \leq n^{1/\gamma}$.
The results of [MS3] also suggest the following problem.

**Problem 2.4.** Formulate and prove a central limit theorem or Donsker’s invariance principle for sufficiently regular fractals with scaling structure where the second moment condition replaced by a condition on γ-moment. Here γ denotes the escape time exponent for sub-Gaussian estimate as before.

3 Stabilizability of Divisible Sandpiles

In a joint work [LMPU] with Levine, Peres and Ugurcan, we study the stabilizability of divisible sandpiles. Consider a connected, locally finite graph \( G = (V,E) \) equipped with the graph Laplacian \( \Delta \), where \( \Delta u(x) = \sum_{y \sim x} (u(y) - u(x)) \). Instead of describing the divisible sandpile model and what it means for an initial configuration \( s : V \to \mathbb{R} \) to be stabilizable, I will introduce an equivalent formulation that avoids such description.

We are interested in the question “Given a divisible sandpile initial configuration \( s : V \to \mathbb{R} \), is \( s \) stabilizable?” Using a least action principle the above question can be reformulated as a question in potential theory: “Given a function \( s : V \to \mathbb{R} \) on a graph \( (V,E) \) does there exists a non-negative function \( u : V \to \mathbb{R} \) such that \( s + \Delta u \leq 1 \) point-wise?” This question is the main focus of our work and it may be of independent interest.

The following theme repeats at several places: The question of stabilizability not only depends on the initial configuration but also the underlying graph. The following Proposition serves as a first illustration of this theme.

**Proposition 3.1.** Let \( G = (V,E) \) be a connected, infinite, locally finite graph and let \( o \in V \) be an arbitrary vertex. Then the divisible sandpile configuration \( 1 + \delta_o \) is stabilizable if and only if the simple random walk on \( G \) is transient.

Our main result answers the question of stabilizability for i.i.d. initial configurations. We show that i.i.d. initial configuration \( s \) is stabilizable almost surely if \( \mathbb{E}s < 1 \) and not stabilizable if \( \mathbb{E}s > 1 \). Therefore there exists a critical density when \( \mathbb{E}s = 1 \). One of our main results is the behavior of i.i.d. sandpile configurations at critical density.

**Theorem 3.2.** Let \( s \) be an i.i.d. divisible sandpile on an infinite, vertex-transitive graph \( G \), with \( \mathbb{E}s = 1 \) and \( 0 < \text{Var}(s) < \infty \). Then \( \mathbb{P}(s \text{ is stabilizable}) = 0 \).

This is in sharp contrast to the abelian sandpile model where such a critical density does not exist (see [FLP10, Proposition 1.4] or [FMR09, Theorem 3.1]). See [FMR09] for more on stabilizability of abelian sandpile model.

Our theme that ‘stabilizability depends on the underlying graph’ repeats itself again in the proof of Theorem 3.2. The proof of Theorem 3.2 splits into three cases depending on the underlying graph. The cases in increasing order of difficulty are

- recurrent; Examples: \( \mathbb{Z}, \mathbb{Z}^2 \)
- transient with \( \sum g(o,x)^2 = \infty \); Examples: \( \mathbb{Z}^3, \mathbb{Z}^4 \)
- transient with \( \sum g(o,x)^2 < \infty \); Examples: \( \mathbb{Z}^d \) with \( d \geq 5 \).

where \( g \) denotes the Green’s function. Other results of [LMPU] include a quantitative version of Theorem 3.2 in \( \mathbb{Z}^d_n \) which involves estimates on Green’s function (killed at a point) on \( \mathbb{Z}^d_n \) using Fourier analysis and assorted examples of stabilizable and not stabilizable configurations.
3.1 Future Directions

In connection to [LMPU], I plan to investigate the following problems in the future.

**Problem 3.3** (Tests for stabilizability; question posed by L. Levine). *Given a divisible sandpile configuration $s : \mathbb{Z}^d \to \mathbb{R}$, find criteria that can distinguish between stabilizable and not stabilizable $s$."

The analogous (and harder) question for abelian sandpiles is also open. A key ingredient in the proof of Theorem 3.2 in [LMPU] is that a random harmonic function on a vertex transitive graph with finite mean and automorphism invariant law is almost surely constant. If we remove the finite mean condition it turns out that there are interesting non constant random harmonic functions with automorphism invariant law. In an ongoing work with Holroyd and Levine, we plan to investigate the following questions.

**Problem 3.4.** *What kind of groups (or more generally vertex-transitive graphs) have non-constant random harmonic functions with automorphism invariant law? If they exist, what can be said about such harmonic functions?"

This problem has connections to boundary theory of random walks (Martin and Poisson boundaries).

References


