A bridge between elliptic and parabolic Harnack inequalities

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Abstract

The notion of conformal walk dimension serves as a bridge between elliptic and parabolic Harnack inequalities. The importance of this notion is due to the fact that the finiteness of the conformal walk dimension characterizes the elliptic Harnack inequality. Roughly speaking, the conformal walk dimension is the infimum of all possible values of the walk dimension that can be attained by a time-change of the process and by a quasisymmetric change of the metric. We show that the conformal walk dimension of any space satisfying the elliptic Harnack inequality is two. We also provide examples that show that the infimum in the definition of conformal walk dimension may or may not be attained.

1 Introduction

What is the ‘best’ way to parametrize a space? This vaguely stated question is the motivation for our work and several earlier works. By a parametrization, we mean a bijection $f: X \to M$ between the given space $X$ and another ‘model space’ $M$ with more desirable properties. For example, the Riemann mapping theorem (or more generally, the uniformization theorem for Riemann surfaces) and geometric flows like the Ricci flow can be viewed as an attempt to answer the above question. In the Riemann mapping theorem example, $X$ is a proper simply connected domain, $M$ is the unit disk, and $f$ is a conformal map. In the Ricci flow example, $X$ is manifold, $M$ is a manifold with constant Ricci curvature, and $f$ is a diffeomorphism. This work aims to formulate and answer this question for spaces satisfying Harnack inequalities. In this work, $X$ is a space that satisfies the elliptic Harnack inequality, $M$ satisfies the stronger parabolic Harnack inequality and $f$ is a quasisymmetry along with a time change of Markov process (quasisymmetry is an analogue of conformal maps for metric spaces).

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This paper uses quasiconformal geometry and time change of Markov process to understand the relationship between elliptic and parabolic Harnack inequalities. The analysis using quasiconformal geometry also leads to a natural uniformization problem for spaces satisfying the elliptic Harnack inequality. Our results can be viewed as a bridge between in analysis in smooth and fractal spaces and also as a bridge between elliptic and parabolic Harnack inequalities.

We informally describe the setup and results. A more precise treatment is given in §2. The setup of this work is a metric measure space equipped with a $m$-symmetric diffusion process, where $m$ is a Radon measure with full support. Equivalently, we consider a metric space $(X,d)$ equipped with a strongly local, regular, Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(X,m)$. We call $(X,d,m,\mathcal{E},\mathcal{F})$ the metric measure space with a strongly local, regular Dirichlet form or MMD space for short. Associated to an MMD space $(X,d,m,\mathcal{E},\mathcal{F})$ is a non-negative self-adjoint operator $L$ on $L^2(m)$ such that the corresponding Markov semigroup $(P_t)_{t\geq0}$ is given by $P_t = e^{-tL}$. The operator $L$ is called the generator of $(X,d,m,\mathcal{E},\mathcal{F})$ is an analogue of the Laplace operator in the abstract setting of MMD spaces. We refer to [FOT, CF] for the theory of Dirichlet forms.

We recall that this setup includes Brownian motion on a Riemannian manifold, where $d$ is the Riemannian distance function, $m$ is the Riemannian measure, $\mathcal{F}$ is the Sobolev space $W^{1,2}$, and $\mathcal{E}(f,f) = \int |\nabla f|^2 dm$, where $\nabla$ denotes the Riemannian gradient. In this case, the corresponding generator $L$ is the Laplace-Beltrami operator with a minus sign (so that $L$ is non-negative definite operator). This setup also covers non-smooth settings like diffusions on fractals including the Sierpinski gasket and Sierpinski carpet. We refer the reader to [Bar98] for an introduction to diffusions on fractals. Random walks on graphs can also be studied in this framework because the corresponding cable process shares many properties with random walk (see [BB04] for this approach).

An MMD space has an associated sheaf of harmonic and caloric functions. Roughly speaking, harmonic functions and caloric functions are generalization of solutions to the ‘Laplace equation’ $\Delta h \equiv 0$ and the ‘heat equation’ $u_t - \Delta u \equiv 0$ respectively. Let $L$ denote the generator of an MMD space $(X,d,m,\mathcal{E},\mathcal{F})$. Let $h : U \to \mathbb{R}$ be a measurable function in an open set $U$. We say that $h$ is harmonic in $U$, if it satisfies $Lh \equiv 0$ in $U$ interpreted in a weak sense. Similarly, we say a space-time function $u : (a,b) \times U \to \mathbb{R}$ is caloric in $(a,b) \times U$ if it satisfies the ‘heat equation’ $\partial_t u + Lu \equiv 0$ interpreted in a weak sense.

Harnack inequalities are fundamental regularity estimates that have numerous applications in partial differential equations and probability theory. We refer to [Kas] for a nice survey on Harnack inequality and its variants. We recall the (scale-invariant) elliptic and parabolic Harnack inequalities. We say that an MMD space $(X,d,m,\mathcal{E},\mathcal{F})$ satisfies the elliptic Harnack inequality (abbreviated as EHI), if there exists $C > 1, \delta \in (0,1)$ such that for all $x \in X, r > 0$ and for any non-negative harmonic function $h$ on the ball $B(x,r)$, we have

$$\text{ess sup}_{B(x,\delta r)} h \leq C \text{ess inf}_{B(x,\delta r)} h.$$  \hspace{1cm} \text{EHI}

We say that an MMD space $(X,d,m,\mathcal{E},\mathcal{F})$ satisfies the parabolic Harnack inequality with walk dimension $\beta$ (abbreviated as PHI($\beta$)), if there exists $0 < C_1 < C_2 < C_3 < C_4 <$
∞, C_5 > 0, δ ∈ (0, 1) such that for all x ∈ X, r > 0 and for any non-negative caloric function u on the space-time cylinder Q = (a, a + C_4 r^β) × B(x, r), we have

$$\text{ess sup}_{Q_-} u \leq C_5 \text{ess inf}_{Q_+} u, \quad \text{PHI(β)}$$

where Q_- = (a + C_1 r^β, a + C_2 r^β) × B(x, δr), and Q_+ = (a + C_3 r^β, a + C_4 r^β) × B(x, δr).

Note that every harmonic function lifts to a caloric function. More precisely, if h is harmonic on B(x, r), then u(t, x) = h(x) is caloric on (a, b) × B(x, r) for all b > a. This lift immediately shows that

$$\text{PHI(β)} \implies \text{EHI}, \quad \text{for all } β > 0. \quad (1.1)$$

However, the converse of the above implication fails. In particular, Delmotte constructs an example of space that satisfies EHI but fails to satisfy PHI(β) for any β > 0 [Del] (see also [BCM]). Nevertheless, one can characterize the elliptic Harnack inequality in terms of the parabolic Harnack inequality [BM, BCM].

The main idea behind the characterization of EHI is to reparametrize the space and time of the diffusion process corresponding to an MMD space so that it satisfies PHI(β) for some β > 0. In the theory of symmetric Dirichlet forms the Revuz correspondence provides a bijection between the time changes of the process and the family of smooth measures. Roughly speaking, smooth measures are Radon measures that do charge any set of capacity zero. If µ is a smooth measure for an MMD space (X, d, m, E, F), then it defines a ‘time changed’ process and an MMD space (X, d, m, E^µ, F^µ). We say a measure µ is admissible, if µ it is a smooth measure and has full quasi-support for the Dirichlet form (E, F) (see Definition 2.9). We denote the collection of admissible measures by A(X, d, m, E, F). Next, we recall the definition of conformal gauge.

**Definition 1.1 (Conformal gauge).** Let (X, d) be a metric space and θ be another metric on X. We say that d and θ are quasisymmetric, if there exists a homeomorphism η : [0, ∞) → [0, ∞) such that

$$\frac{θ(x, a)}{θ(x, b)} \leq η \left( \frac{d(x, a)}{d(x, b)} \right) \quad \text{for all triples of points } x, a, b ∈ X, x \neq b.$$  

The conformal gauge of a metric space (X, d) is defined as

$$\mathcal{J}(X, d) := \{θ : X × X → [0, ∞) : θ \text{ is a metric on } X; d \text{ and } θ \text{ are quasisymmetric} \}.$$  

Being quasisymmetric is an equivalence relation among metrics. That is,

$$\mathcal{J}(X, d) = \mathcal{J}(X, θ) \quad \text{for all } θ ∈ \mathcal{J}(X, d). \quad (1.2)$$

The notion of quasisymmetry is a generalization of conformal map to the context of metric spaces. This is the reason behind the terminology ‘conformal gauge’. We refer to [Hei, HK] for an exposition to the theory of quasisymmetric maps and quasiconformal geometry on metric spaces.
To characterize the elliptic Harnack inequality, we reparametrize the space by choosing a new metric in the conformal gauge of $(X, d)$ and we reparametrize time by choosing a new symmetric measure that is admissible. More precisely, given an MMD space $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfying the EHI, we seek to find a metric $\theta \in J(X, d)$ and a measure $\mu \in A(X, d, m, \mathcal{E}, \mathcal{F})$ such that the corresponding time-changed MMD space $(X, \theta, \mu, \mathcal{E}_\mu, \mathcal{F}_\mu)$ equipped with the new metric $\theta$ satisfies PHI($\beta$) for some $\beta > 0$. In other words, we seek to upgrade from EHI to PHI($\beta$) by reparametrizing space and time. This motivates the notion of conformal walk dimension.

**Definition 1.2.** The conformal walk dimension $d_{cw}$ of an MMD space $(X, d, m, \mathcal{E}, \mathcal{F})$ is defined as

$$d_{cw} = \inf \left\{ \beta > 0 \mid \text{there exists } \mu \in A(X, d, m, \mathcal{E}, \mathcal{F}) \text{ and } \theta \in J(X, d) \text{ such that } (X, \theta, \mu, \mathcal{E}_\mu, \mathcal{F}_\mu) \text{ satisfies PHI}(\beta) \right\},$$

where $\inf \emptyset = +\infty$ and $(\mathcal{E}_\mu, \mathcal{F}_\mu)$ denotes the time-changed Dirichlet form on $L^2(X, \mu)$.

Note that if $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfies PHI($\beta$), then it is easy to see that for any $r \in (0, 1]$ the MMD space $(X, d^r, m, \mathcal{E}, \mathcal{F})$ satisfies PHI($\beta/r$) and $d^r \in J(X, d)$. This shows that it is easy to increase the walk dimension by changing to a different metric in the conformal gauge but it is non-trivial to decrease the walk dimension. This explains the ‘infimum’ in (1.3).

Two natural questions arise. What is the value of $d_{cw}$? When is the infimum in (1.3) attained? The answer to the first question is given below. We will always assume that our metric space $(X, d)$ is complete, separable, locally compact and satisfies the metric doubling property. Recall that a metric space $(X, d)$ satisfies the metric doubling property, if there exists $N \in \mathbb{N}$ such that for all $x \in X, r > 0$, the ball $B(x, r)$ can be covered by $N$ balls of radii $r/2$.

Our first main result is that the value of the conformal walk dimension is always two for any space satisfying the elliptic Harnack inequality. This allows us to sharpen the existing characterization of the elliptic Harnack inequality. Let $d_{cw}$ denote the conformal walk dimension of $(X, d, m, \mathcal{E}, \mathcal{F})$. Then the following are equivalent:

(a) $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfies the elliptic Harnack inequality.

(b) $d_{cw} < \infty$.

(c) $d_{cw} = 2$.

The equivalence between (a) and (b) is contained in [BM, BCM]. That (c) implies (b) is obvious. Our contribution to the above equivalence is (a) implies (c). Therefore our result sharpens the characterization of EHI in [BM, BCM]. The result (a) implies (c) is particularly interesting on fractals as we explain below. Diffusions on many regular fractals are known to satisfy PHI($\beta$) with $\beta > 2$. These are often called anomalous diffusions to distinguish from the classical smooth settings like the Euclidean space where one often
has Gaussian space-time scaling and PHI(2). However, by the above equivalence one can ‘improve’ from PHI($\beta$) to PHI($2 + \epsilon$) for any $\epsilon > 0$ even on fractals. So this result serves a bridge between anomalous space-time scaling in fractals and Gaussian space-time scaling seen in smooth settings.

It is worth mentioning that the proof of (a) implies (b) in [BM, BCM] do not give an universal upper bound for $d_{cw}$. The bound on $d_{cw}$ obtained depends on the constants in EHI and could be arbitrarily large. To improve from the previous (a) implies (b) result to (a) implies (c), we need a new construction of metric and measures.

We briefly discuss this new construction in the proof of (a) implies (c). The inspiration behind our argument is the uniformization theorem for Riemann surfaces. In the proof of the uniformization theorem, the Green’s function of a Riemann surface (or a subset of the surface) plays an essential role in constructing the uniformizing map [Mar19, Chapter 15]. We use certain cutoff functions across annuli with small Dirichlet energy at different scales and locations as a substitute for the Green’s function. It is helpful to think of these cutoff functions as equilibrium potentials across annuli. Roughly speaking, the diameter of a ball under the new metric $\theta \in J(X,d)$ for our construction is proportional to the average gradient of equilibrium potential chosen at suitable location and scale.

On a technical level, our proof relies heavily on the theory of Gromov hyperbolic spaces. We view $X$ as the boundary of a Gromov hyperbolic space called the hyperbolic filling. The conformal gauge of $X$ is essentially in a bijective correspondence to bi-Lipschitz change of metrics on the hyperbolic filling. The bi-Lipschitz change of metric on the hyperbolic filling is constructed using equilibrium potentials as described above. A major ingredient in the proof is a combinatorial description of the conformal gauge due to Carrasco Piaggio [Car13].

Theorem 2.10 is a partial converse to the trivial implication PHI($\beta$) $\implies$ EHI. The equivalence between (a) and (c) in Theorem 2.10, clarifies the extent to which the converse of the implication in (1.1) holds. Although the value of $d_{cw}$ has a simple description, the following questions remain open in general.

**Problem 1.3.** (1) (Attainment problem) Given an MMD space $(X, d, m, \mathcal{E}, \mathcal{F})$ that satisfies the elliptic Harnack inequality, is the infimum in (1.3) is attained?

(2) (Gaussian uniformization problem) Given an MMD space $(X, d, m, \mathcal{E}, \mathcal{F})$ that satisfies the elliptic Harnack inequality, describe all metrics $\theta \in \mathcal{J}(X,d)$ and measures $\mu \in \mathcal{A}(X,d,m,\mathcal{E},\mathcal{F})$ such that the corresponding time-changed MMD space $(X, \theta, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu)$ satisfies PHI(2).

We describe two examples of self-similar fractals for which a positive answer to the attainment problem is known. Kigami shows that Brownian motion on two dimensional Sierpinski gasket attains the infimum, where $\mu$ is the Kusuoka measure and $\theta$ is the corresponding intrinsic metric [Kig08]. Further examples of admissible measures that attain the infimum for the two dimensional Sierpinski carpet is described in [Kaj12]. In retrospect, Kigami’s measurable Riemannian structure on the Sierpinski gasket is the first evidence towards the implication (a) $\implies$ (c) in Theorem 2.10. Another example
of a fractal that attains the infimum in (1.3) is the two dimensional snowball described in [Mur19]. The ‘snowball’ fractal can be viewed as a limit of Riemann surfaces and is a higher dimensional analogue of the von Koch snowflake. In this example, the answer to the attainment problem is obtained by considering a limit of uniformizing maps to $S^2$ and using the conformal invariance of Brownian motion. Our terminology ‘Gaussian uniformization problem’ is inspired by this example and the fact that PHI(2) is equivalent to Gaussian heat kernel estimates.

Nevertheless, infimum in (1.3) need not be attained in general. We show in §6.3 that Vicsek set and three dimensional Sierpinski gasket do not attain the infimum in (1.3). We do not know the answer to the attainment problem for the standard (two dimensional) Sierpinski carpet. The examples with non-attainment of $d_{cw}$ rely on the following result. For a ‘regular’ fractal, if the infimum in (1.3) is attained then it is possible to choose $\mu \in A(X, d, m, E, F)$ as the energy measure of a function that is harmonic outside a canonical boundary. This result immediately implies the non-attainment of $d_{cw}$ for the Vicsek tree, since energy measure of any such harmonic function does not have full quasi-support. The non-attainment of $d_{cw}$ for three-dimensional Sierpinski gasket requires a more delicate analysis of the intrinsic metric corresponding to energy measure.

Next, we mention some progress towards the Gaussian uniformization problem. If $(X, \theta, \mu, E^\mu, F^\mu)$ satisfies PHI(2), then by the results in [KM], we know that the metric $\theta$ is determined by the measure $\mu$ up to a bi-Lipschitz change. In this case, the metric $\theta$ is bi-Lipschitz equivalent to the intrinsic metric. Therefore, in order to find a metric $\theta \in J(X, d)$ and a measure $\mu \in A(X, d, m, E, F)$ in the Gaussian uniformization problem, it is enough to find an appropriate measure $\mu$. Furthermore, by [KM], we know that the $\mu$ is a minimal energy dominant measure. Since any two minimal energy dominant measures are mutually absolutely continuous, any two admissible measures that arise in the Gaussian uniformization problem are mutually absolutely continuous. We strengthen this result by showing that any two admissible measures that arise in the Gaussian uniformization problem are $A_{\infty}$-related in $(X, d)$ in the sense of Muckenhoupt.

The result on $A_{\infty}$-relation between admissible measures in the Gaussian uniformization problem and its proof are inspired by a similar result for Ahlfors regular conformal dimension on Loewner spaces [HK, Theorem 7.11]. The combinatorial description of conformal gauge used in the proof of Theorem 2.10 was developed for studying Ahlfors regular conformal dimension in [Car13]. Therefore, we find it appropriate to recall the definition of Ahlfors regular conformal dimension and discuss some related questions.

Given a metric space $(X, d)$ and a Borel measure $\mu$, we say that $\mu$ is $p$-Ahlfors regular if there exists $C > 0$ such that

$$C^{-1} r^p \leq \mu(B(x, r)) \leq C r^p$$

for all $x \in X, r > 0$ such that $B(x, r) \neq X$.

It is easy to verify that if a $p$-Ahlfors regular measure $\mu$ exists on $(X, d)$, then the $p$-dimensional Hausdorff measure $\mathcal{H}_p$ is also $p$-Ahlfors regular and the Hausdorff dimension of $(X, d)$ is $p$. Therefore, the existence of a $p$-Ahlfors regular measure is a property of the
metric $d$. The Ahlfors regular conformal dimension of a metric space $(X,d)$ is defined as

$$d_{ARC}(X,d) = \left\{ p > 0 \mid \text{there exists } \theta \in \mathcal{J}(X,d) \text{ and a } p\text{-Ahlfors regular measure } \mu \text{ on } (X,\theta) \right\}. \quad (1.4)$$

The attainment problem for the Ahlfors regular conformal dimension of the standard (two dimensional) Sierpinski carpet is a well-known open question [BK05, Problem 6.2]. An important motivation for studying the attainment problem is Cannon’s conjecture in geometric group theory. Cannon’s conjecture states that every finitely generated, Gromov-hyperbolic group $G$ whose boundary (in the sense of Gromov) is homeomorphic to the 2-sphere is a Kleinian group. Bonk and Kleiner show that Cannon’s conjecture is equivalent to the attainment of Ahlfors regular conformal dimension of the boundary of such a group [BK05, Theorem 1.1]. Our results and proof techniques will make it clear that there are similarities between the attainment problems for Ahlfors regular conformal dimension and the conformal walk dimension. We hope that some of the methods we develop towards Gaussian uniformization problem has applications to the attainment problem for Ahlfors regular conformal dimension.

Our work suggests that it would be useful to develop a theory of non-linear Dirichlet forms to study Ahlfors regular conformal dimension on fractals. In particular, Theorem 6.16 shows that if the infimum in (1.3) is attained on a self similar fractal, then an optimal admissible measure can be chosen to be the energy measure of a harmonic function. This result and its proof suggest that an optimal Ahlfors regular measure attaining the Ahlfors regular conformal dimension can be constructed as the ‘energy measure’ of a $p$-harmonic function. However, the notion of energy measure for non-linear Dirichlet energy remains to be developed on fractals (non-linear Dirichlet energy can be formally viewed as $\int |\nabla f|^p$ with $p \neq 2$). There is a well-developed non-linear potential theory in smooth settings (see [HKM] and references there) but a similar theory is yet to be developed on fractals.

## 2 Framework and results

In this section, we recall the background definitions and state our main results.

### 2.1 Metric measure Dirichlet space and energy measure

Throughout this paper, we consider a complete, locally compact separable metric space $(X,d)$, equipped with a Radon measure $m$ with full support, i.e., a Borel measure $m$ on $X$ which is finite on any compact subset of $X$ and strictly positive on any non-empty open subset of $X$. Such a triple $(X,d,m)$ is referred to as a metric measure space.

Furthermore let $(\mathcal{E}, \mathcal{F})$ be a symmetric Dirichlet form on $L^2(X,m)$; by definition, $\mathcal{F}$ is a dense linear subspace of $L^2(X,m)$, and $\mathcal{E} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ is a non-negative definite symmetric bilinear form which is closed ($\mathcal{F}$ is a Hilbert space under the inner product $\mathcal{E}_1 := \mathcal{E} + \langle \cdot, \cdot \rangle_{L^2(X,m)}$) and Markovian ($f^+ \land 1 \in \mathcal{F}$ and $\mathcal{E}(f^+ \land 1, f^+ \land 1) \leq \mathcal{E}(f,f)$ for any
functions, sup \subset f for a Borel measurable function \in F, lim_{n \to \infty} E local, regular symmetric Dirichlet form (continuous. The pair (X, d, m, \nu, \sigma) of a metric measure space \sigma \geq 1 is called a minimal energy-dominant measure if the following two conditions are satisfied:

(i) (Domination) For every \nu \in F, \Gamma(f, f) \ll \nu.

(ii) (Minimality) If another \sigma-finite Borel measure \nu' on X satisfies condition (i) with \nu replaced by \nu', then \nu \ll \nu'.

Note that by [Hin10, Lemmas 2.2, 2.3 and 2.4], a minimal energy-dominant measure of (E, F) always exists. By the minimality property, we note that any two minimal energy dominant measures are mutually absolutely continuous.

We recall the definition of intrinsic metric

Definition 2.3. Let (X, d, m, E, F) be an MMD space. We define its intrinsic metric \(d_{int} : X \times X \to [0, \infty] \) by

\[ d_{int}(x, y) := \sup\{f(x) - f(y) \mid f \in F_{loc} \cap C(X), \Gamma(f, f) \leq m\}, \tag{2.2} \]
where
\[ \mathcal{F}_{\text{loc}} := \left\{ f \mid f \text{ is an } m\text{-equivalence class of } \mathbb{R}\text{-valued Borel measurable functions on } X \text{ such that } f1_V = f^#1_V \text{ m-a.e. for some } f^# \in \mathcal{F} \text{ for each relatively compact open subset } V \text{ of } X \right\} \] (2.3)

and the energy measure \( \Gamma(f, f) \) of \( f \in \mathcal{F}_{\text{loc}} \) associated with \((X, d, m, \mathcal{E}, \mathcal{F})\) is defined as the unique Borel measure on \( X \) such that \( \Gamma(f, f)(A) = \Gamma(f^#, f^#)(A) \) for any relatively compact Borel subset \( A \) of \( X \) and any \( V, f^# \) as in (2.3) with \( A \subset V \); note that \( \Gamma(f^#, f^#)(A) \) is independent of a particular choice of such \( V, f^# \).

### 2.2 Harnack inequalities

We recall the definition of harmonic and caloric functions.

**Definition 2.4.** Let \((X, d, m, \mathcal{E}, \mathcal{F})\) be an MMD space and let \( \mathcal{F}_e \) extended Dirichlet space corresponding to \((X, d, m, \mathcal{E}, \mathcal{F})\) [FOT, p. 40]. A function \( h \in \mathcal{F}_e \) is said to be \( \mathcal{E} \)-harmonic on an open subset \( U \) of \( X \), and
\[ \mathcal{E}(h, f) = 0 \] (2.4)
for all \( f \in \mathcal{F} \cap C_c(X) \) with \( \text{supp}_m[f] \subset U \).

Let \( I \) be an open interval in \( \mathbb{R} \). We say that a function \( u : I \to L^2(X, m) \) is weakly differentiable at \( t_0 \in I \) if for any \( f \in L^2(X, m) \) the function \( t \mapsto \langle u(t), f \rangle \) is differentiable at \( t_0 \), where \( \langle \cdot, \cdot \rangle \) denotes the inner product in \( L^2(X, m) \). By the uniform boundedness principle, there exists a (unique) function \( w \in L^2(X, m) \) such that
\[ \lim_{t \to t_0} \left\langle \frac{u(t) - u(t_0)}{t - t_0}, f \right\rangle = \langle w, f \rangle, \quad \text{for all } f \in L^2(X, m). \]
We say that the function \( w \) is the weak derivative of the function \( u \) at \( t_0 \) and write \( w = u'(t_0) \).

Let \( I \) be an open interval in \( \mathbb{R} \) and let \( \Omega \) be an open subset of \( X \). A function \( u : I \to \mathcal{F} \) is said to be caloric in \( I \times \Omega \) if \( u \) is weakly differentiable in the space \( L^2(\Omega) \) at any \( t \in I \), and for any \( f \in C_c(\Omega) \cap \mathcal{F} \), and for any \( t \in I \),
\[ \langle u', f \rangle + \mathcal{E}(u, f) = 0. \] (2.5)

**Definition 2.5** (Harnack inequalities). We say that an MMD space \((X, d, m, \mathcal{E}, \mathcal{F})\) satisfies the **elliptic Harnack inequality** (abbreviated as EHI), if there exists \( C > 1, \delta \in (0, 1) \) such that for all \( x \in X, r > 0 \) and for any non-negative harmonic function \( h \in \mathcal{F}_e \) on the ball \( B(x, r) \), we have
\[ \text{ess sup}_{B(x, \delta r)} h \leq C \text{ ess inf}_{B(x, \delta r)} h. \] EHI

We say that an MMD space \((X, d, m, \mathcal{E}, \mathcal{F})\) satisfies the **parabolic Harnack inequality** with walk dimension \( \beta \) (abbreviated as PHI(\( \beta \))), if there exists \( 0 < C_1 < C_2 < C_3 < C_4 < \)
Consider an MMD space \((X,d,m,\mathcal{E},\mathcal{F})\). Given a Borel set \(A\), we define its 1-capacity as
\[
\text{Cap}_1(A) = \inf \{ \mathcal{E}(f,f) + \|f\|_2^2 : f \in \mathcal{C}(X) \cap \mathcal{F} \text{ such that } f \equiv 1 \text{ on a neighborhood of } A \},
\]
where \(\|f\|_2\) denotes the \(L^2(X,m)\) norm. For disjoint Borel sets \(A,B\) such that \(B\) is closed and \(A \subseteq B^c\) (by \(A \subseteq B^c\), we mean that \(A\) is compact and \(A \subset B^c\)), we define \(\mathcal{F}(A,B)\) as the set of function \(\phi \in \mathcal{F}\) such that \(\phi \equiv 1\) in an open neighborhood of \(A\), and \(\text{supp } \phi \subset B^c\). For such sets \(A\) and \(B\), we define the capacity between them as
\[
\text{Cap}(A,B) = \inf \{ \mathcal{E}(f,f) : f \in \mathcal{F}(A,B) \}.
\]

**Definition 2.6** (Smooth measures). Let \((X,d,m,\mathcal{E},\mathcal{F})\) be an MMD space. A Radon measure \(\mu\) on \(X\) is smooth if \(\mu\) charges no set of zero capacity (that is \(\text{Cap}_1(A) = 0\) implies \(\mu(A) = 0\)).

An increasing sequence \(\{F_k : k \geq 1\}\) of compact subsets of a MMD space \((X,d,m,\mathcal{E},\mathcal{F})\) is said to be a nest if \(\bigcup_{k \geq 1} \mathcal{F}_{F_k}\) is \(\sqrt{\mathcal{E}_1}\)-dense in \(\mathcal{F}\), where \(\mathcal{F}_{F_k} := \{ f \in \mathcal{F} : f = 0 \text{ m-a.e. on } X \setminus F_k \}\) and \(\mathcal{E}_1(f,f) = \mathcal{E}(f,f) + \|f\|_2^2\). Recall that \(D \subset X\) is quasi open if there exists a nest \(\{F_n\}\) such that \(D \cap F_n\) is an open subset of \(F_n\) in the relative topology for each \(n \in \mathbb{N}\). The complement of a quasi open set is called quasi closed. We recall the definition of quasi support of a smooth measure \(\mu\) [FOT, p. 190].

**Definition 2.7** (Full quasi support). Let \(\mu\) be a smooth measure of an MMD space \((X,d,m,\mathcal{E},\mathcal{F})\). We say that \(\mu\) has full quasi support if for any quasi closed set \(F\) such that \(\mu(X \setminus F) = 0\), we have \(\text{Cap}_1(X \setminus F) = 0\).

**Definition 2.8** (Time changed Dirichlet form). If \(\mu\) is a smooth Radon measure, it defines a time change of the process whose associated Dirichlet form is called the trace Dirichlet form (denoted by \((\mathcal{E}^\mu,\mathcal{F}^\mu)\)) [CF, Section 5.2] and [FOT, Section 6.2]. Let \(\mathcal{F}_e\) denote the extended Dirichlet space corresponding to \((X,d,m,\mathcal{E},\mathcal{F})\) [FOT, p. 40]. If \(\mu\) has full quasi support, then the trace Dirichlet form \((\mathcal{E}^\mu,\mathcal{F}^\mu)\) is given by
\[
\mathcal{F}^\mu = \mathcal{F}_e \cap L^2(X,\mu), \quad \mathcal{E}^\mu(u,u) = \mathcal{E}(u,u), \quad \text{for } u \in \mathcal{F}^\mu.
\]

By [FOT, Theorem 5.1.5 and Theorem 6.2.1], \((\mathcal{E}^\mu,\mathcal{F}^\mu)\) is a regular Dirichlet form on \(L^2(X,\mu)\). In probabilistic terms, the time changed process is \((\omega,t) \mapsto Y_{\tau_t(\omega)}(\omega)\), where \(\tau_t\) is the right continuous inverse of the positive continuous additive functional \(A_t\) such that the Revuz measure of \(A_t\) is \(\mu\), where \((Y_t)_{t \geq 0}\) is the diffusion associated with \((X,d,m,\mathcal{E},\mathcal{F})\).
We are interested in the class of admissible measures which are smooth measures with full quasi-support.

**Definition 2.9.** [Admissible measures] Let \((X, d, m, \mathcal{E}, \mathcal{F})\) be an MMD space. We say that a Borel measure \(\mu\) on \(X\) is admissible if \(\mu\) is a smooth Radon measure with full quasi-support. We denote the class of admissible measures of an MMD space by \(\mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F})\).

We note that if \(\mu \in \mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F})\), then by [CF, Theorem 5.2.11]

\[
\mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F}) = \mathcal{A}(X, d, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu). \tag{2.7}
\]

### 2.4 Main results

Our first main result is that the value of the conformal walk dimension is an invariant for spaces satisfying the EHI. We recall that the conformal walk dimension of a MMD space \((X, d, m, \mathcal{E}, \mathcal{F})\) is the infimum of all \(\beta > 0\) such that there exists an admissible measure \(\mu \in \mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F})\) and a metric \(\theta \in \mathcal{J}(X, d)\) such that the time changed MMD space \((X, \theta, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu)\) satisfies the parabolic Harnack inequality \(\text{PHI}(\beta)\).

**Theorem 2.10** (Universality of conformal walk dimension). Let \((X, d, m, \mathcal{E}, \mathcal{F})\) be an MMD space such that \((X, d)\) is a complete, separable, locally compact space that satisfies the metric doubling property. Let \(d_{cw}\) denote the conformal walk dimension of \((X, d, m, \mathcal{E}, \mathcal{F})\). Then the following are equivalent:

(a) \((X, d, m, \mathcal{E}, \mathcal{F})\) satisfies the elliptic Harnack inequality.

(b) \(d_{cw} < \infty\).

(c) \(d_{cw} = 2\).

The next question is whether or not the infimum in the definition of \(d_{cw}\) in (1.3) is attained. To this end we first describe the metric and measure. The following result is essentially contained in [KM].

**Proposition 2.11.** Let \((X, d, m, \mathcal{E}, \mathcal{F})\) be an MMD space that satisfies \(\text{PHI}(2)\).

(a) Then the metric \(d\) is bi-Lipschitz equivalent to the intrinsic metric \(d_{int}\).

(b) The symmetric measure \(m\) is a minimal energy-dominant measure.

By Proposition 2.11(a), in order to find a metric \(\theta \in \mathcal{J}(X, d)\) and a measure \(\mu \in \mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F})\) in the Gaussian uniformization problem, it is enough to find an appropriate measure \(\mu\) as the metric \(\theta\) is essentially determined by the measure up to a bi-Lipschitz transform. Since constructing measures are typically easier than constructing metrics, it is useful to restrict our attention to finding suitable measures in the Gaussian uniformization problem. By Proposition 2.11(b), any such measure is determined uniquely up to a mutually absolutely continuous change of measure. In fact, we have the following improvement of Proposition 2.11. Any two such measures \(\mu_1, \mu_2\) that attain the infimum in the definition of \(d_{cw}\) are \(A_\infty\)-related in the sense of Muckenhoupt (see Definition 5.4).
Theorem 2.12. Let \((X, d, m, \mathcal{E}, \mathcal{F})\) be an MMD space that satisfies the elliptic Harnack inequality. Let \(\left( X, \theta_i, \mu_i, \mathcal{E}^{\mu_i}, \mathcal{F}^{\mu_i} \right), i = 1, 2 \) be two time-changed MMD spaces that satisfy PH\(I(2)\) such that \(\theta_i \in \mathcal{J}(X, d), \mu_i \in \mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F})\) for \(i = 1, 2\). Then the measures \(\mu_1\) and \(\mu_2\) are \(A_\infty\)-related in \((X, d)\).

Using Proposition 2.11, Theorem 2.12 and sharp constants of Poincaré inequalities in [CW], we answer the Gaussian uniformization problem for Brownian motion on \(\mathbb{R}\). We do not know the answer to Gaussian uniformization problem for Brownian motion in higher dimensional Euclidean space \(\mathbb{R}^n, n \geq 2\).

2.5 Outline to the proof of Theorem 2.10

By changing metric and measure, we may assume that \((X, d, m, \mathcal{E}, \mathcal{F})\) satisfies PH\(I(\gamma)\) for some \(\gamma > 2\). Let \(\beta > 2\) be arbitrary. We wish to construct a metric \(\theta \in \mathcal{J}(X, d)\) and measure \(\mu \in \mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F})\) such that the trace Dirichlet form \((X, \theta, \mu, \mathcal{E}^{\mu}, \mathcal{F}^{\mu})\) satisfies PH\(I(\beta)\).

To sketch the main ideas, we further assume that \((X, d)\) is compact and the diameter of \((X, d)\) is normalized to \(\frac{1}{3}\). The non-compact case follows by the same argument as the compact case, by considering \(X\) as a limit of compact sets. Using known characterizations of the PH\(I(\beta)\), it is enough to construct a metric \(\theta \in \mathcal{J}(X, d)\) and a measure \(\mu \in \mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F})\) such that

\[
\mu \left( B_{\theta}(x, r) \right) \asymp r^\beta \text{Cap} \left( B_{\theta}(x, r), B_{\theta}(x, 2r) \right), \quad \text{for all } x \in X, r \lesssim \text{diam}(X, \theta). \tag{2.8}
\]

The above estimate relating the measure \(\mu\) and capacity implies that \(\mu\) is a smooth measure with full quasi support and satisfies the following volume doubling and reverse volume doubling properties (see Proposition 4.11): there exists \(C_D > 0\) and \(c_D \in (0, 1)\) such that

\[
\frac{\mu(B_{\theta}(x, 2r))}{\mu(B_{\theta}(x, r))} \leq C_D, \quad \text{for all } x \in X, r > 0, \tag{2.9}
\]

\[
\frac{\mu(B_{\theta}(x, r))}{\mu(B_{\theta}(x, 2r))} \geq c_D, \quad \text{for all } x \in X, 0 < r \lesssim \text{diam}(X, d). \tag{2.10}
\]

The estimate (2.8) along with Theorem 4.4, implies PH\(I(\beta)\) because volume doubling, reverse volume doubling, and EHI are preserved by the quasisymmetric change of the metric from \(d\) to \(\theta\).

The construction of metric \(\theta\) and measure \(\mu\) is a modification of [Car13], but instead of the Ahlfors regularity required in [Car13] we need to establish (2.8). Following [Car13], we construct the metric \(\theta\) and measure \(\mu\) that satisfy (2.8) using a multi scale argument. This part of the argument relies on theory of Gromov hyperbolic spaces. The basic idea behind the approach is to construct a graph (called the hyperbolic filling) that is Gromov hyperbolic, whose boundary (in the sense of Gromov) corresponds to the given metric space \((X, d)\). A well-known result in Gromov hyperbolic spaces asserts that any metric in the conformal gauge \(\mathcal{J}(X, d)\) up to bi-Lipschitz equivalence can be obtained by a bounded
perturbation of edge weights on the hyperbolic filling. We recall the basic results about Gromov hyperbolic spaces and their boundaries in §3.1.

We first sketch the construction of this hyperbolic space postponing a more precise definition to §3.2. We choose a parameter $a > 10^2$ and cover the space $X$ using a covering $\mathcal{S}_n$ with balls of radii $2a^{-n}$ such that for any two distinct balls $B(x_1, 2a^{-n}), B(x_2, 2a^{-n}) \in \mathcal{S}_n$, we have $B(x_1, a^{-n}/2) \cap B(x_1, a^{-n}/2) = \emptyset$ (we think of these balls as 'approximately pairwise disjoint'). Therefore the covering $\mathcal{S}_n$ corresponds to scale $a^{-n}$ for all $n \in \mathbb{N}_{\geq 0}$. In what follows, for a ball $B$, we denote by $x_B$ and $r_B$ the center and radius of $B$. For a ball $B$ and $\lambda > 0$, we denote by $\lambda \cdot B$, the ball $B(x_B, \lambda r_B)$.

We define a tree of vertical edges with vertex set $\coprod_{n \geq 0} \mathcal{S}_n$ by choosing for each ball $B \in \mathcal{S}_n, n \geq 1$ a ‘parent ball’ $B' \in \mathcal{S}_{n-1}$ such that $x_B$ is a closest point to $x_{B'}$ in the set $\{x_C : C \in \mathcal{S}_{n-1}\}$. By the assumption on the diameter, we may assume that $\mathcal{S}_0$ is a singleton. The edges in this tree are called vertical edges. We choose another parameter $\lambda \geq 10$ such to define another set of edges on $\coprod_{n \geq 0} \mathcal{S}_n$ called the horizontal edge. Two distinct balls $B, \tilde{B} \in \mathcal{S}_n, n \geq 0$ share a horizontal edge if and only if $\lambda \cdot B \cap \lambda \cdot \tilde{B} \neq \emptyset$. The edges of the hyperbolic filling is the union of horizontal and vertical edges.

In our construction, the vertical edge weights play a more central role and the value of horizontal edge weights are less important. The weight of the vertical edge between $B \in \mathcal{S}_n, B' \in \mathcal{S}_{n-1}$ can be interpreted as the relative diameter under the $\theta$-metric. More precisely, let us define the relative diameter of $B \in \mathcal{S}_n, n \geq 1$ as

$$\rho(B) := \frac{\text{diam}(B, \theta)}{\text{diam}(B', \theta)},$$

(2.11)

where $B' \in \mathcal{S}_{n-1}$ is such that there is a vertical edge between $B'$ and $B$ ($B'$ is the parent of $B$ in the tree of vertical edges). It turns out that the ‘relative diameter’ in (2.11) contains enough information about $\theta$ to reconstruct the metric $\theta$ (up to bi-Lipschitz equivalence). So, we could reduce the problem of construction of $\theta \in \mathcal{J}(X,d)$ to constructing the function $\rho(\cdot)$ on $\coprod_{n \geq 1} \mathcal{S}_n$; see Theorem 3.14.

It is therefore enough to construct $\rho(\cdot)$ in (2.11). Next, we describe two key conditions that the relative diameter $\rho(\cdot)$ defined in (2.11) must satisfy. For a ball $B \in \mathcal{S}_{n-1}, n \geq 1$, let us denote by $\Gamma_n(B)$ the set of horizontal paths in $\mathcal{S}_n$ defined by

$$\Gamma_n(B) = \left\{ (B_i)_{i=0}^N \mid N \in \mathbb{N}, B_i \in \mathcal{S}_n \text{ for all } i = 0, \ldots, N; \ B_i \text{ and } B_{i+1} \text{ share a horizontal edge for all } i = 0, \ldots, N-1, \ x_{B_0} \in B, \ x_B \notin 2 \cdot B \right\}.$$

The first condition on $\rho(\cdot)$ is

$$\sum_{i=1}^N \rho(B_i) \geq 1, \quad \text{for all } (B_i)_{i=0}^N \in \Gamma_n(B) \text{ and } B \in \mathcal{S}_{n-1}, n \geq 1. \quad (2.12)$$

The condition (2.12) is a consequence of the fact that $\theta \in \mathcal{J}(X,d)$ and that $(X,d)$ is a uniformly perfect metric space. We like to think of (2.12) as a 'no shortcuts condition' as it disallows the possibility of short cuts in the $\theta$-metric from $B_d(x, r)$ to $B_d(x, 2r)^c$. 


The second condition arises from the estimate (2.8). For any ball $B \in \mathcal{S}_k$, $k \geq 0$, by $\mathcal{D}_{k+1}(B)$, we denote its descendants in $\mathcal{S}_{k+1}$; that is $D_{k+1}(B)$ is the set of elements $B' \in \mathcal{S}_{k+1}$ such that $B'$ and $B$ share a vertical edge. The second condition is that $\rho$ must satisfy is the following estimate:

$$
\sum_{B' \in \mathcal{D}_{k+1}(B)} \rho(B') \beta \text{Cap}(B', (2 \cdot B')^c) \lesssim \text{Cap}(B, (2 \cdot B)^c), \quad \text{for all } B \in \mathcal{S}_k \text{ and } k \text{ large enough.} 
$$

(2.13)

To explain (2.13), we first observe that the volume doubling property (2.9) implies that

$$
\sum_{B' \in \mathcal{D}_{k+1}(B)} \mu(B') \lesssim \mu(B), \quad \text{for all } B \in \mathcal{S}_k.
$$

By (2.8), $\theta \in \mathcal{J}(X,d)$ and comparing estimates using EHI, one obtains $\mu(B) \asymp \text{diam}(B, \theta)^\beta \text{Cap}(B, (2 \cdot B)^c)$ for all $B \in \mathcal{S}_k$ and for all large enough $k$. Combining these estimates and with (2.9), we obtain (2.13). To summarize, the conditions (2.12) and (2.13) arise from the metric and measure respectively. It turns out that the necessary conditions (2.12) and (2.13) on $\rho(\cdot)$ are ‘almost sufficient’ to construct $\rho$; see Theorem 3.24.

We note that there is a tension between the estimates (2.12) and (2.13). On the one hand, in order to satisfy (2.12), the function $\rho(\cdot)$ must be large enough, whereas (2.13) imposes that $\rho(\cdot)$ can not to too large. Next, we sketch how to construct $\rho$ that satisfies these seemingly conflicting requirements in (2.12) and (2.13). Let $B \in \mathcal{S}_{k+1}$, $u \in C_c(X)$ be such that

$$
u \equiv 1 \text{ on } B(x_B, 1.1 r_B), \quad u \equiv 0 \text{ on } B(x_B, 1.9 r_B)^c. \quad \mathcal{E}(u,u) \asymp \frac{m(B)}{r_B^\gamma}. \quad (2.14)
$$

Such a function exists because of (2.8) for $(X,d,m,\mathcal{E},\mathcal{F})$ with $\beta$ replaced by $\gamma$ and a covering argument. It is helpful to think of $u$ as the equilibrium potential corresponding to $\text{Cap}(B(x_B, 1.1 r_B), B(x_B, 1.9 r_B)^c)$. Let us define the functions $u_B, \rho_B : \mathcal{S}_{k+1} \to [0, \infty)$ as

$$
u_B(B') := \int_{B'} u \, dm = \frac{1}{m(B')} \int_{B'} u \, dm, \\
\rho_B(B'') = \sum_{B'' \in \mathcal{S}_{k+1}, B'' \sim B'} |u_B(B'') - u_B(B')|,$$

where $B'' \sim B'$ means that $B''$ and $B'$ share a horizontal edge (or equivalently, $\lambda \cdot B'' \cap \lambda \cdot B' \neq \emptyset$). From the definitions it is clear that $u_B$ is a discrete version of $u$ and $\rho_B$ is a discrete version of the gradient of $u$. Using a Poincaré inequality on $(X,d,m,\mathcal{E},\mathcal{F})$ and the bound $\text{Cap}(B', (2 \cdot B')^c) \asymp \frac{m(B')}{r_{B'}}$, we obtain the following estimate (see (4.18)):

$$
\sum_{B' \in \mathcal{D}_{k+1}(B)} \rho_B(B')^2 \text{Cap}(B', (2 \cdot B')^c) \lesssim \mathcal{E}(u,u) \lesssim \text{Cap}(B, (2 \cdot B)^c). \quad (2.15)
$$
Since $u_B(B_0) = 1$ for any $B_0 \in \mathcal{S}_{k+1}$ with $x_{B_0} \in B$ and $u_B(B_N) = 0$ for any $B_N \in \mathcal{S}_{k+1}$ with $x_{B_N} \notin (2 \cdot B)^c$, by the triangle inequality

$$\sum_{i=1}^N \rho_B(B_i) \geq 1, \quad \text{for all } (B_i)_{i=0}^N \in \Gamma_{k+1}(B). \quad (2.16)$$

Clearly, (2.16) is a local version of (2.12) and (2.15) is a version of (2.13) with $\beta = 2$. Here, by ‘local version’ we mean that the estimates (2.12) and (2.13) are ensures for a fixed value of $B$. To ensure (2.12) and (2.13) for all scales and locations, we define

$$\rho(B') = \sup_{B \in \mathcal{S}_k} \rho_B(B'), \quad \text{for all } B' \in \mathcal{S}_{k+1}, \quad (2.17)$$

where $\rho_B$ is defined as above at all locations and scales. This ensures (2.12) and (2.13) at all scales with $\beta = 2$.

However, $\rho$ should satisfy further additional conditions that the above construction need not obey. Since $\theta \in \mathcal{J}(X, d)$, one obtains that

$$\rho(B) \gtrsim 1 \quad \text{for all } B \in \mathcal{S}_k, k \geq 1. \quad (2.18)$$

However, (2.17) need not satisfy (2.18). This requires us to increase $\rho$ further if necessary. We define the ‘diameter function’

$$\pi(B) = \prod_{k=0}^n \pi(B_i), \quad \text{for all } B \in \mathcal{S}_n, \quad (2.19)$$

where $B_i \in \mathcal{S}_i$ for all $i = 0, \ldots, n$ and $B_n = B$ and there is a vertical edge between $B_i$ and $B_{i+1}$ for all $i = 0, \ldots, n - 1$. If $\rho$ is given by (2.11), then clearly $\pi(B) = \text{diam}(B, \theta)/\text{diam}(X, \theta)$. By quasisymmetry, $\text{diam}(B, \theta) \approx \text{diam}(B', \theta)$ whenever $B$ and $B'$ share a horizontal edge. This suggests the following condition on $\rho$:

$$\pi(B) \approx \pi(B'), \quad \text{whenever } B \text{ and } B' \text{ share a horizontal edge}, \quad (2.20)$$

where $\pi$ is defined as given in (2.19). Similarly, for constructing measure, we need to ensure that $\rho$ satisfies

$$\pi(B)^\beta \text{Cap}(B, (2 \cdot B)^c) \approx \sum_{B' \in \mathcal{D}_n(B)} \pi(B')^\beta \text{Cap}(B', (2 \cdot B')^c), \quad \text{for all } B \in \mathcal{S}_k \text{ and } n > k \geq 0, \quad (2.21)$$

where $\mathcal{D}_n(B)$ denotes the descendants of $B$ in $\mathcal{S}_n$. The conditions (2.20) and (2.21) are rather delicate because the value of $\pi$ can change drastically if we change $\rho$ by a bounded multiplicative factor. This is due to the fact that the multiplicative ‘errors’ in $\rho$ accumulate as we move to finer and finer scales. This suggests that we need to control the constants very carefully.

To achieve this we need to consider $\beta > 2$ (instead of $\beta = 2$ considered above). We choose $\rho$ defined in (2.17) uniformly small by picking a function $u$ that satisfies (2.14)
along with an additional scale invariant Hölder continuity estimate (see (4.23), (4.24) and (4.25)). Then using

$$\sum_{B' \in \mathcal{D}_{k+1}(B)} \rho_B(B')^\beta \text{Cap}(B', (2 \cdot B')^c) \leq \|\rho\|_\infty^{\beta - 2} \sum_{B' \in \mathcal{D}_{k+1}(B)} \rho_B(B') \text{Cap}(B', (2 \cdot B')^c)$$

$$\lesssim \|\rho\|_\infty^{\beta - 2} \text{Cap}(B, (2 \cdot B)^c),$$

we obtain enough control on the constants in (2.13) to ensure (2.18) and (2.20) after further modification of \( \rho \). By the Hölder continuity estimate on \( u \), \( \|\rho\|_\infty \) can be made arbitrarily small by increasing the vertical parameter \( a \).

3 Metric and measure via hyperbolic filling

Given a metric space, it is often useful to view the space as the boundary of a Gromov hyperbolic space. Such a viewpoint is prevalent but often implicit in various multi-scale arguments in analysis and probability. Roughly speaking, a metric space viewed simultaneously at different locations and scales has a natural hyperbolic structure. A nice introduction to this viewpoint can be found in [Sem01]. This will be made precise by the notion of hyperbolic filling in §3.2. The main tool for the construction of metric is Theorem 3.14(a), and the construction of measure uses Lemma 3.20. To describe the construction, we recall the definition of hyperbolic space in the sense of Gromov.

3.1 Gromov hyperbolic spaces and their boundary

We briefly recall the basics of Gromov hyperbolic spaces and refer the reader to [CDP, GH, Gro, Vä05] for a detailed exposition.

Let \( (Z, d) \) be a metric space. Given three points \( x, y, p \in Z \), we define the Gromov product of \( x \) and \( y \) with respect to the base point \( w \) as

\[
(x|y)_w = \frac{1}{2}(d(x, w) + d(y, w) - d(x, y)).
\]

By the triangle inequality, Gromov product is always non-negative. We say that a metric space \( (Z, d) \) is \( \delta \)-hyperbolic, if for any four points \( x, y, z, w \in Z \), we have

\[
(x|z)_w \geq (x|y)_w \land (y|z)_w - \delta.
\]

We say that \( (Z, d) \) is hyperbolic (or \( d \) is a hyperbolic metric), if \((Z, d)\) is hyperbolic for some \( \delta \in [0, \infty) \). If the above condition is satisfied for a fixed base point \( w \in Z \), and arbitrary \( x, y, z \in Z \), then \((Z, d)\) is \( 2\delta \)-hyperbolic [CDP, Proposition 1.2].

Next, we recall the notion of the boundary of a hyperbolic space. Let \((Z, d)\) be a hyperbolic space and let \( w \in Z \). A sequence of points \( \{x_i\} \subset Z \) is said to converge at infinity, if

\[
\lim_{i,j \to \infty} (x_i|x_j)_w = \infty.
\]
The above notion of convergence at infinity does not depend on the choice of the base point \( w \in Z \), because by the triangle inequality \(|(x|y)_w - (x|y)_w'| \leq d(w, w')\).

Two sequences \( \{x_i\}, \{y_i\} \) that converge at infinity are said to be \textit{equivalent}, if

\[
\lim_{i \to \infty} (x_i|y_i)_w = \infty.
\]

This defines an equivalence relation among all sequences that converge at infinity. As before, it is easy to check that the notion of equivalent sequences does not depend on the choice of the base point \( w \). The \textit{boundary} \( \partial Z \) of \((Z, d)\) is defined as the set of equivalence classes of sequences converging at infinity under the above equivalence relation. If there are multiple hyperbolic metrics on the same set \( Z \), to avoid confusion, we denote the boundary of \((Z, d)\) by \( \partial (Z, d) \) (see Lemma 3.13(d)). The notion of Gromov product can be defined on the boundary as follows: for all \( a, b \in \partial Z \)

\[
(a|b)_w = \sup \left\{ \lim \inf_{i \to \infty} (x_i|y_i)_w : \{x_i\} \in a, \{y_i\} \in b \right\},
\]

and similarly, for \( a \in \partial Z, y \in Z \), we define

\[
(a|y)_w = \sup \left\{ \lim \inf_{i \to \infty} (x_i|y)_w : \{x_i\} \in a \right\}.
\]

The boundary \( \partial Z \) of the hyperbolic space \((Z, d)\) carries a family of metrics. Let \( w \in Z \) be a base point. A metric \( \rho : \partial Z \times \partial Z \to [0, \infty) \) on \( \partial Z \) is said to be a \textit{visual metric} with \textit{visual parameter} \( \alpha > 1 \) if there exists \( k_1, k_2 > 0 \) such that

\[
k_1\alpha^{-(a|b)_w} \leq \rho(a, b) \leq k_2\alpha^{-(a|b)_w}.
\]

Note that visual metrics depend on the choice of the base point, and on the visual parameter \( \alpha \). If a visual metric with base point \( w \) and visual parameter \( \alpha \) exists, then it can be chosen to be

\[
\rho_{\alpha,w}(a, b) := \inf \sum_{i=1}^{n-1} \alpha^{-(a_i|a_{i+1})_w},
\]

where the infimum is over all finite sequences \( \{a_i\}_{i=1}^n \subset \partial Z, n \geq 2 \) such that \( a_1 = a, a_n = b \).

Visual metrics exist as we recall now. A metric space \((Z, d)\) is said to be \textit{proper} if all closed balls are compact. For any \( \delta \)-hyperbolic space \((Z, d)\), there exists \( \alpha_0 > 1 \) (\( \alpha \) depends only on \( \delta \)) such that if \( \alpha \in (1, \alpha_0) \), then there exists a visual metric with parameter \( \alpha \) [GH, Ch. 7], [BS, Lemma 6.1]. It is well-known that quasi-isometry between almost geodesic hyperbolic spaces induces a quasisymmetry on their boundaries. Since this plays a central role in our construction of metric, we recall the relevant definitions and results below.

We say that a map (not necessarily continuous) \( f : (X_1, d_1) \to (X_2, d_2) \) between two metric spaces is a \textit{quasi-isometry} if there exists constants \( A, B > 0 \) such that

\[
A^{-1}d_1(x, y) - A \leq d_2(f(x), f(y)) \leq Ad_1(x, y) + A,
\]

for all \( x, y \in X_1 \), and

\[
\sup_{x_2 \in X_2} d(x_2, f(X_1)) = \sup_{x_2 \in X_2} \inf_{x_1 \in X_1} d(x_2, f(x_1)) \leq B.
\]
Definition 3.1. A distortion function is a homeomorphism of $[0, \infty)$ onto itself. Let $\eta$ be a distortion function. A map $f : (X_1, d_1) \to (X_2, d_2)$ between metric spaces is said to be $\eta$-quasisymmetric, if $f$ is a homeomorphism and

$$\frac{d_2(f(x), f(a))}{d_2(f(x), f(b))} \leq \eta \left( \frac{d_1(x, a)}{d_1(x, b)} \right)$$

for all triples of points $x, a, b \in X_1, x \neq b$. We say $f$ is a quasisymmetry if it is $\eta$-quasisymmetric for some distortion function $\eta$. We say that metric spaces $(X_1, d_1)$ and $(X_2, d_2)$ are quasisymmetric, if there exists a quasisymmetry $f : (X_1, d_1) \to (X_2, d_2)$. We say that metrics $d_1$ and $d_2$ on $X$ are quasisymmetric (or, $d_1$ is quasisymmetric to $d_2$), if the identity map $\text{Id} : (X, d_1) \to (X, d_2)$ is a quasisymmetry. A quasisymmetry $f : (X_1, d_1) \to (X_2, d_2)$ is said to be a power quasisymmetry, if there exists $\alpha > 0, \lambda \geq 1$ such that $f$ is $\eta_{\alpha, \lambda}$-quasisymmetric, where

$$\eta_{\alpha, \lambda}(t) = \begin{cases} \lambda t^{1/\alpha}, & \text{if } 0 \leq t < 1, \\ \lambda t^\alpha, & \text{if } t \geq 1. \end{cases}$$

Recall that the conformal gauge of a metric space $(X, d)$ is defined as

$$\mathcal{J}(X, d) := \{ \theta : X \times X \to [0, \infty) : \theta \text{ is a metric on } X, \text{ and } \theta \text{ is quasisymmetric to } d \}$$

Bi-Lipschitz maps are the simplest examples of quasi-symmetric maps. Recall that a map $f : (X_1, d_1) \to (X_2, d_2)$ is said to be bi-Lipschitz, if there exists $C \geq 1,$

$$C^{-1}d_1(x, y) \leq d_2(f(x), f(y)) \leq Cd_1(x, y), \quad \text{for all } x, y \in X_1.$$ 

Two metrics $d_1, d_2 : X \times X \to [0, \infty)$ on $X$ are said to be bi-Lipschitz equivalent if the identity map $\text{Id} : (X, d_1) \to (X, d_2)$ is bi-Lipschitz.

We collect a few useful facts about quasisymmetric maps.

Proposition 3.2. ([MT10, Lemma 1.2.18] and [Hei, Proposition 10.8]) Let the identity map $\text{Id} : (X, d_1) \to (X, d_2)$ be an $\eta$-quasisymmetry for some distortion function $\eta$. By $B_i(x, r)$ we denote the open ball in $(X, d_i)$ with center $x$ and radius $r > 0$, for $i = 1, 2$.

(a) For all $A \geq 1, x \in X, r > 0$, there exists $s > 0$ such that

$$B_2(x, s) \subset B_1(x, r) \subset B_1(x, Ar) \subset B_2(x, \eta(A)s).$$

In (3.1), $s$ can be defined as

$$s = \sup \{0 \leq s_2 < 2\text{diam}(X, d_2) : B_2(x, s_2) \subset B_1(x, r)\}$$

Conversely, for all $A > 1, x \in X, r > 0$, there exists $t > 0$ such that

$$B_1(x, r) \subset B_2(x, t) \subset B_2(x, At) \subset B_1(x, A_1r),$$

where $A_1 = \zeta(A)$ and $\zeta(t)$ is the distortion function given by $\zeta(t) = 1/\eta(t^{-1})$. In (3.3), $t$ can be defined as

$$At = \sup \{0 \leq r_2 < 2\text{diam}(X, d_2) : B_2(x, Ar_2) \subset B_1(x, A_1r)\}.$$
(b) If $A \subset B \subset X$ such that $0 < \text{diam}(A,d_1) \leq \text{diam}(B,d_1) < \infty$, then $0 < \text{diam}(A,d_2) \leq \text{diam}(B,d_2) < \infty$ and

\[
\frac{1}{2\eta \left( \frac{\text{diam}(B,d_1)}{\text{diam}(A,d_1)} \right)} \leq \text{diam}(A,d_2) \leq \eta \left( \frac{2\text{diam}(A,d_1)}{\text{diam}(B,d_1)} \right).
\]

(3.5)

**Definition 3.3.** A metric space $(X,d)$ is $k$-almost geodesic, if for every $x,y \in X$ and every $t \in [0,d(x,y)]$, there is some $z \in X$ with $|d(x,z) - t| \leq k$ and $|d(y,z) - (d(x,y) - t)| \leq k$. We say that a metric space is almost geodesic if it is $k$-almost geodesic for some $k \geq 0$. We recall that quasi-isometry between almost geodesic hyperbolic spaces induces a quasisymmetry between their boundaries.

**Proposition 3.4.** ([BS, Theorem 6.5 and Proposition 6.3]) Let $(Z_1,d_1)$ and $(Z_2,d_2)$ be two almost geodesic, $\delta$-hyperbolic metric spaces. Let $f : (Z_1,d_1) \to (Z_2,d_2)$ denote quasi-isometry.

(a) If $\{x_i\} \subset Z_1$ converges at infinity, then $\{f(x_i)\} \subset Y$ converges at infinity. If $\{x_i\}$ and $\{y_i\}$ are equivalent sequences in $X$ converging at infinity, then $\{f(x_i)\}$ and $\{f(y_i)\}$ are also equivalent.

(b) If $a \in \partial Z_1$ and $\{x_i\} \subset a$, let $b \in \partial Z_2$ be the equivalence class of $\{f(x_i)\}$. Then $\partial f : \partial Z_1 \to \partial Z_2$ is well-defined, and is a bijection.

(c) Let $p_1 \in Z_1$ be a base point in $Z_1$, and let $f(p_1)$ be a corresponding base point in $Z_2$. Let $\rho_1, \rho_2$ denote visual metrics (with not necessarily the same visual parameter) on $\partial Z_1, \partial Z_2$ with base points $p_1, f(p_1)$ respectively. Then the induced boundary map $\partial f : (\partial Z_1, \rho_1) \to (\partial Z_2, \rho_2)$ is a power quasisymmetry.

**Remark 3.5.** The distortion function $\eta$ for the quasisymmetry $\partial f$ in (c) above can be chosen to depend only on the constants associated with the quasi-isometry $f : Z_1 \to Z_2$ and the constants associated with the properties of being almost geodesic and Gromov hyperbolic for $Z_1, Z_2$.

### 3.2 Hyperbolic filling of a compact metric space

Given a compact metric space $(X,d)$, one can construct an almost geodesic, hyperbolic space whose boundary equipped with a visual metric can be identified with $(X,d)$. We assume further that $(X,d)$ is doubling and uniformly perfect. Recall that a metric space $(X,d)$ is $K_P$-doubling, if any ball $B(x,r)$ can be covered by $K_P$ balls of radius $r/2$. A metric space is $K_P$-uniformly perfect, if for any ball $B(x,r)$ such that $X \setminus B(x,r) \neq \emptyset$, then $B(x,r) \setminus B(x,r/K_P) \neq \emptyset$.

We recall the notion of hyperbolic filling due to Bourdon and Pajot [BP], based on a similar construction due to Elek [Ele]. We recall the definition in [Car13]. Let $(X,d)$ be a compact, doubling, uniformly perfect, metric space. For a ball $B = B(x,r)$ and $\alpha > 0$, by $\alpha \cdot B$ we denote the ball $B(x,\alpha r)$. We fix two parameter $a > 8$ and $\lambda \geq 3$. The
parameters $a$ and $\lambda$ are respectively called the *vertical* and *horizontal* parameters of the hyperbolic filling. For each $n \geq 0$, let $S_n$ denote a finite covering of $X$ by open balls such that for all $B \in S_n$, there exists a center $x_B \in X$ such that

$$B = B(x_B, 2a^{-n}), \quad (3.6)$$

and for any distinct pair $B \neq B'$ in $S_n$, we have

$$B(x_B, a^{-n}/2) \cap B(x_{B'}, a^{-n}/2) = \emptyset. \quad (3.7)$$

We assume that

$$S_0 = \{X\} \quad (3.8)$$

is a singleton (by scaling the metric if necessary). We remark that the assumption (3.8) is just for convenience. For arbitrary (but finite) diameter, we choose $n_0 \in \mathbb{Z}$ such that $a^{-n_0} > \text{diam}(X, d) \geq a^{-n_0-1}$. For the general compact case we replace $0$ with $n_0$, so that we have coverings $S_k$ for all $k \geq n_0$ such that $S_k$ is a covering by ‘almost’ pairwise disjoint balls of radii roughly $a^{-k}$ as given in (3.6) and (3.7).

We construct a graph whose vertex set is $S = \bigsqcup_{n=0}^\infty S_n$. Next, we construct a tree structure of *vertical edges* on $S$. For each $n \geq 0$, we partition $S_{n+1}$ into pairwise disjoint sets $\{T_n(B) : B \in S_n\}$ indexed by $S_n$, with $S_{n+1} = \bigsqcup_{B \in S_n} T_n(B)$ satisfying the following property:

$$\text{if } B' \in T_n(B) \text{ for some } B \in S_n, B' \in S_{n+1}, \text{ then } d(x_{B'}, A_n) = d(x_{B'}, x_B). \quad (3.9)$$

In other words, if $B' \in T_n(B)$, then $x_B \in A_n$ is a minimizer to the distance between $x_{B'}$ and $A_n$. Since such a minimizer always exists, there exists a (not necessarily unique) partition $\{T_n(B) : B \in S_n\}$ of $S_{n+1}$ for all $n \geq 0$. We call the elements of $T_n(B)$ as the *children* of $B$. From now on, let us fix one such partition $\{T_n(B) : B \in S_n\}$ for each $n \geq 0$. We say that there exist a *vertical edge* between two sets $B, B' \in S$, if there exists $n \geq 0$ such that either $B \in S_n, B' \in S_{n+1} \cap T_n(B)$ or $B' \in S_{n+1}, B \in S_{n+1} \cap T_n(B')$; in other words, one of them is a child of the other. Note that the vertical edges form a tree on the vertex set $S$, with base point (or root) $w$, where $S_0 = \{w\}$. The unique path from the base point to a vertex $B \in S$ denotes the *genealogy* $g(B)$. More precisely, we define the genealogy $g(B)$ as (3.6) if $B \in S_0$, and then $g(B)$ is defined as

$$g(B) = \begin{cases} \{B\}, & \text{if } B \in S_0 \\ \{B_0, B_1, \ldots, B_n\}, & \text{if } B = B_n \in S_n, n \geq 1, \text{ and } B_{i+1} \in T(B_i), \text{ for } i = 0, \ldots, n-1. \end{cases}$$

In the above definition, if $0 \leq i < n$, we denote the vertex $B_i \in S_i$ by $g(B)_i$. If $B \in S_n$, and $l > n$, we define $D_l(B)$ as the descendants of $B$ in the generation $l$

$$D_l(B) := \{B' \in S_l : g(B')_n = B\}. \quad (3.10)$$

For $B \in S_n$, we denote $\cup_{l \geq n+1} D_l(B)$ by $D(B)$ which are the descendants of $B$.

Using the horizontal parameter $\lambda \geq 3$, we define another family of edges on the vertex set $S$ call the *horizontal edges*. We say $B \sim B'$ if there exists $n \geq 0$ such that $B, B' \in S_n$ and $\lambda \cdot B \cap \lambda \cdot B' \neq \emptyset$. We say that there is a horizontal edge between $B, B' \in S$, if $B \sim B'$ and they are distinct (so as to avoid self-loops).
Definition 3.6 (Hyperbolic filling). Let $S_d = (S, E)$ denote the graph with vertices in $S$ and whose edges $E$ are obtained by the taking the union of horizontal and vertical edges. With a slight abuse of notation, we often view $S_d$ as a metric space equipped with the (combinatorial) graph distance, which we denote by $D_S : S \times S \to \mathbb{Z}_{\geq 0}$. The metric space $S_d = (S, D_S)$ is almost geodesic and hyperbolic [BP, Proposition 2.1]. The metric space $S_d$ is said to be a hyperbolic filling of $(X, d)$.

We refer to §3.3 for a construction of hyperbolic filling. Note that the hyperbolic filling is not unique as we make an arbitrary choice of covering. Even if the covering is fixed, the choice of children $T_n(B)$ is not necessarily unique. Nevertheless, any two hyperbolic fillings (with possibly different parameters) of a metric space are quasi-isometric to each other [BP, Corollaire 2.4].

We fix the base point of $S_d$ to be $w \in S$, where $\{w\} = S_0$. We now define a map $p : X \to \partial S_d$ that identifies $X$ with the boundary of $S_d$ as follows. For each $x \in X$, choose a sequence $\{B_i\}$ with $x \in B_i \in S_i$, $i \in \mathbb{N}$. Then it is easy to see that the sequence $\{B_i\}$ converges at infinity. Let $p(x) \in \partial S_d$ denote the equivalence class containing $\{B_i\}$.

The map $p$ is a bijection and its inverse $p^{-1} : \partial S_d \to X$ can be described as follows. For any $a \in \partial S_d$, and for any $\{B_i\} \in a$, the corresponding sequence of centers $\{x_{B_i}\}$ is a convergent sequence in $X$, and the limit is $p^{-1}(a) = \lim_{i \to \infty} x_{B_i}$. The map $p^{-1}$ is well-defined; that is, if $\{B_i\}$ and $\{B_i'\}$ are equivalent sequences that converge at infinity, then $\lim_{i \to \infty} x_{B_i} = \lim_{i \to \infty} x_{B_i'}$.

We summarize the properties of the hyperbolic filling $S_d$ and its boundary $\partial S_d$ as follows:

Proposition 3.7. ([BP, Proposition 2.1]) Let $(X, d)$ denote a compact, doubling, uniformly perfect metric space. Let $S_d$ denote a hyperbolic filling with vertical parameter $a > 1$, and horizontal parameter $\lambda \geq 3$. Then $S_d$ is almost geodesic, Gromov hyperbolic space. The map $p : X \to \partial S_d$ is a homeomorphism between $X$ and $\partial S_d$. If we choose the base point $w \in S_d$ as the unique vertex in $S_0$, then there exists $K > 1$ such that

$$K^{-1} a^{-\left(\rho(x) \rho(y)\right)_w} \leq d(x, y) \leq Ka^{-\left(\rho(x) \rho(y)\right)_w}$$

for all $x, y \in X$.

By the above proposition we can recover the metric space $(X, d)$ from its hyperbolic filling $S_d$ with horizontal parameter $\lambda$ and vertical parameter $a$ (up to bi-Lipschitz equivalence) as the boundary $\partial S_d$ equipped with a visual metric with base point $w$ and visual parameter $a$.

For technical reasons following [Car13, (2.8)], we will often assume that

$$\lambda \geq 32, \quad a \geq 24(\lambda \vee K_P),$$

where $K_P$ is such that $(X, d)$ is $K_P$-uniformly perfect.
3.3 Construction of hyperbolic fillings

Since the metric spaces we deal with need not be compact, we need a suitable substitute for hyperbolic fillings. To circumvent this difficulty, we view the metric space as an increasing union of compact spaces and construct a sequence of hyperbolic fillings. Quasisymmetric maps and doubling measures have nice compactness properties that persist under such limits.

We recall the notion of net in a metric space.

**Definition 3.8.** Let \((X, d)\) be a metric space and let \(\varepsilon > 0\). A subset \(N \subseteq X\) is called an \(\varepsilon\)-net in \((X, d)\) if the following two conditions are satisfied:

1. (Separation) \(N\) is \(\varepsilon\)-separated in \((X, d)\), i.e., \(d(x, y) \geq \varepsilon\) for any \(x, y \in N\) with \(x \neq y\).
2. (Maximality) If \(N \subseteq M \subseteq X\) and \(M\) is \(\varepsilon\)-separated in \((X, d)\), then \(M = N\).

In the lemma below, we recall a standard construction of hyperbolic filling and some of its properties.

**Lemma 3.9.** (Cf. [Car13, Lemma 2.2] and [KRS, Theorem 2.1]) Let \((X, d)\) be a complete, \(K_P\)-uniformly perfect, \(K_D\)-doubling metric space such that either \(\text{diam}(X, d) = \frac{1}{2}\) or \(\infty\). Let \(a > 8\) and let \(x_0 \in X\). Let \(N_0\) be a 1-net in \((X, d)\) such that \(x_0 \in N_0\). Define inductively the sets \(N_k\) for \(k \in \mathbb{N}\) such that

\[
N_{k-1} \subset N_k, \text{ and } N_k \text{ is a } a^{-k}\text{-net in } (X, d), \text{ for all } k \in \mathbb{N}.
\]

For \(k < 0\) and \(k \in \mathbb{Z}\), we define \(N_k\) to be a \(a^{-k}\)-net in \((N_{k+1}, d)\) such that \(x_0 \in N_k\) for all \(k \in \mathbb{Z}\) (Note that \(N_k\) need not be a \(a^{-k}\)-net in \((X, d)\) for \(k < 0\)). For each \(x \in N_k\) and \(k \in \mathbb{Z}\), we pick a predecessor \(y \in N_{k-1}\) such that \(y\) is a closest point to \(x\) in \(N_{k-1}\) (by making a choice if there is more than one closest point); that is \(y \in N_{k-1}\) satisfies

\[
d(x, y) = \min_{z \in N_{k-1}} d(x, z).
\]

For any \(x \in N_k, k \in \mathbb{Z}\), we denote its predecessor as defined above by \(P(x) \in N_{k-1}\).

(a) For all \(k \in \mathbb{Z}\), and for any two distinct points \(x, y \in N_k\), we have

\[
B(x, a^{-k}/2) \cap B(y, a^{-k}/2) = \emptyset.
\] (3.12)

We have the following covering property:

\[
\bigcup_{x \in N_k} B(x, a^{-k}) = X, \text{ for all } k \geq 0,
\] (3.13)

\[
\bigcup_{x \in N_k} B(x, (1 - a^{-1})^{-1}a^{-k}) = X, \text{ for all } k \in \mathbb{Z}.
\] (3.14)

In particular, if \(\text{diam}(X, d) = \frac{1}{2}\), the coverings \(S_n = \{B(y, (1 - a^{-1})^{-1}a^{-n}) \mid y \in N_n\}\) for all \(n \geq 0\) is a covering that satisfies (3.6) and (3.7). For any \(n \geq 0\) and for any \(B = B(x_B, a^{-n}) \in S_n\), the sets

\[
T_n(B) = \{B(y, a^{-n-1}) \mid y \in N_{k+1} \text{such that } x_B = P(y)\}
\]

forms a partition of \(S_{n+1}\) as required by (3.9).

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(b) Let $a, \lambda$ satisfy \((3.11)\). Let $y \leq N_{k+1}$ be such that

$$B(y, (1 - a^{-1})^{-1}a^{-k-1}) \cap B(P(y), a^{-k}/3) \neq \emptyset.$$  

Then for any $z \in N_{k+1}$ such that $d(y, z) < (1 - a^{-1})^{-1}a^{-k-1}$, we have $P(y) = P(z)$. (In other words, $y$ corresponds to the center of a non-peripheral ball in $S_{k+1}$ as given in Definition 3.23).

(c) Let $k \in \mathbb{Z}$ and $y \in N_k$. Let $D_k(y)$ denote the set of descendants of $y$ defined by

$$D_k(y) = \{y\} \cup \{z \in N_l \mid \text{such that } l > k \text{ and } P^{l-k}(z) = y\}. \quad (3.15)$$

Then

$$\overline{B}(y, (1 - a^{-1})^{-1}a^{-k}) \supset D_k(y) \supset B(y, (2^{-1} - (a - 1)^{-1})a^{-k}). \quad (3.16)$$

The space $\overline{D_k(y)}$ with the restricted metric $d$ is $K_p^2$-doubling and $K'_p$-uniformly perfect, where $K'_p = 2aK_P(1 - a^{-1})^{-1}(2^{-1} - (a - 1)^{-1})^{-1}$.

Proof.

(a) The properties \((3.12)\) and \((3.13)\) follow from the separation and maximality properties of the $a^{-k}$-net $N_k$ in $X$ respectively. We use the notation $P^k(y)$ denote the $k$-predecessor of $y$ (for example, $P^2(y) = P(P(y))$). To show \((3.14)\), by \((3.13)\) it suffices to consider the case $k < 0$. By \((3.13)\), for any $y \in X$ there exists $y_0 \in N_0$ such that $d(y_0, y) < 1$. Define $y_l = P^{-l}(y)$ for all $l < 0$. Since $d(y_l, y_{l+1}) < a^{-l}$ for all $l < 0$, we have

$$d(y_k, y) \leq d(y_0, y) + \sum_{l=1}^{k} d(y_l, y_{l+1}) < \sum_{l=0}^{k} a^{-l} = (1 - a^{-1})^{-1}(a^{-k} - a^{-1}) < (1 - a^{-1})^{-1}a^{-k}. \quad (3.17)$$

Since $y \in X$ is arbitrary and $y_k \in N_k$, we have \((3.14)\).

(b) By the triangle inequality, we have

$$d(z, P(y)) \leq d(z, y) + d(y, P(y)) \leq (2\lambda + 1)(1 - a^{-1})^{-1}a^{-k-1} + \frac{1}{3}a^{-k}$$

$$< a^{-k}/2 \quad \text{(by \((3.11)\))}$$

By \((3.13)\) and $d(z, P(y)) < a^{-k}/2$, we conclude that $P(z) = P(y)$.

(c) By \((3.13)\) and triangle inequality, we have

$$d(y, z) \geq a^{-k}/2, \quad \text{for all } z \in N_{k+1} \setminus D(y) \text{ and for all } y \in N_k. \quad (3.18)$$

By \((3.17)\), we have

$$\overline{D(z)} \subset \overline{B}(z, (1 - a^{-1})^{-1}a^{-k-1}), \quad \text{for all } z \in N_{k+1}. \quad (3.19)$$
Since $\bigcup_{w \in N_l} \overline{D(w)}$ is dense and closed (by the doubling property), we have
\[
\bigcup_{w \in N_l} \overline{D(w)} = X, \quad \text{for all } l \in \mathbb{Z}. \tag{3.20}
\]

Combining (3.18), (3.19), (3.20) and using triangle inequality, we obtain (3.16).

Next, we show that $\overline{D(y)}$ is $K_D$-doubling. More generally, we show that any subset $Y \subset X$ is $K^2_D$-doubling. Let $B(x, r) \cap Y, x \in Y$ be an arbitrary ball in $Y$. Since $(X, d)$ is $K_D$-doubling, the ball $B(x, r)$ can be covered by $N$ balls $B(x_i, r/4), i = 1, \ldots, N$, where $N \geq K^2_D$. If $B(x_i, r/4) \cap Y \neq \emptyset$, we choose $y_i \in B(x_i, r/4) \cap Y$, so that $B(x_i, r/4) \subset B(y_i, r/2)$. Hence all such balls $B(y_i, r/2) \cap Y$ cover $B(x, r) \cap Y$.

Let $B(x, r) \cap \overline{D(y)}$ be an arbitrary ball in $\overline{D(y)}$ such that $x \in \overline{D(y)}$ and $B(x, r) \cap \overline{D(y)} \neq \overline{D(y)}$. Let $n \in \mathbb{Z}$ be the unique integer such that
\[
(1 - a^{-1})^{-1} a^{-n} < r \leq (1 - a^{-1})^{-1} a^{-n+1}.
\]

Since $\overline{D(y)} = \bigcup_{z \in N\cap \overline{D(y)}} \overline{D(z)}$ for all $n \geq k$, by (3.16), there exists $z \in D(y) \cap N_n$ such that
\[
d(z, x) \leq (1 - a^{-1})^{-1} a^{-n} < r. \tag{3.21}
\]

Since $(X, d)$ is $K_p$-uniformly perfect, and using (3.16) and $B(z, (2^{-1} - (a - 1)^{-1}) a^{-n}) \neq X$, there exists $w \in \overline{D(y)}$ such that
\[
(2^{-1} - (a - 1)^{-1})^{-1} a^{-n} > d(w, z) \geq \frac{1}{K_p} (2^{-1} - (a - 1)^{-1}) a^{-n}. \tag{3.22}
\]

We consider two cases, depending on whether or not $d(z, x) < \frac{1}{2} a^{-n}$. If $d(z, y) \geq \frac{1}{2} a^{-n}$, then
\[
r > d(z, x) \geq \frac{1}{2} a^{-n} \geq \frac{r}{2a((1 - a^{-1})^{-1})}. \tag{3.23}
\]

On the other hand, if $d(z, x) < \frac{1}{2} a^{-n}$, then
\[
d(w, x) \lor d(z, x) \leq d(z, w) + d(z, x) < (2^{-1} - (a - 1)^{-1}) a^{-n} + \frac{1}{2} a^{-n} < a^{-n} < r.
\]

Hence by (3.22), if $d(z, x) < \frac{1}{2} a^{-n}$, we have
\[
d(w, x) \lor d(z, x) \geq \frac{1}{2} d(w, z) \geq \frac{1}{2K_p} (2^{-1} - (a - 1)^{-1}) a^{-n}
\[
geq \frac{r}{2aK_p(1 - a^{-1})^{-1}(2^{-1} - (a - 1)^{-1})^{-1}}. \tag{3.24}
\]

By (3.23) and (3.24), $\overline{D(y)}$ is $2aK_p(1 - a^{-1})^{-1}(2^{-1} - (a - 1)^{-1})^{-1}$-uniformly perfect. \hfill \square
**Definition 3.10** (Extended hyperbolic filling). Let \((X, d)\) be a complete, \(K_P\)-uniformly perfect doubling metric space such that either \(\text{diam}(X, d) = \frac{1}{2}\) or \(\infty\). Let \(a > 8\), \(\lambda \geq 32\) be constants that satisfy (3.11). Let \(x_0 \in X\) and consider the sets \(N_k, k \in \mathbb{Z}\) as defined in Lemma 3.9. Define
\[
S_k = \{B(x, 2a^{-k}) : x \in N_k\}, k \in \mathbb{Z}.
\]
For any \(k \in \mathbb{Z}\) and for any pair of distinct balls \(B, B' \in S_k\), we say that there is a **horizontal edge** between \(B\) and \(B'\) (denoted as \(B \sim B'\)) if and only if \(\lambda \cdot B \cap \lambda \cdot B' \neq \emptyset\). For any \(k \in \mathbb{Z}\) and for any \(B(x, 2a^{-k}) \in S_k\), \(B(y, 2a^{-k-1}) \in S_{k+1}\), we say that there is a **vertical edge** between \(B(x, 2a^{-k})\) and \(B(y, 2a^{-k-1})\), if \(x\) is the predecessor of \(y\) (as defined in Lemma 3.9). We define a graph \((V, E)\) with vertex set \(V = \bigsqcup_{k \in \mathbb{Z}} S_k\) and the edge set \(E\) defined by the union of horizontal and vertical edges. This graph is called the **extended hyperbolic filling** of \((X, d)\) with horizontal parameter \(\lambda\) and vertical parameter \(a\).

If \((X, d)\) is compact, the subgraph of the extended hyperbolic filling induced by \(\mathcal{S} = \bigsqcup_{k \in \mathbb{Z} \geq 0} S_k\) forms a hyperbolic filling as given in Definition 3.6.

On the other hand, if \((X, d)\) is non-compact, we view \(X\) as an increasing limit of compact spaces \(\overline{D_l(x_0)}\) as \(l \to -\infty\), where \(D_k(x_0)\) is as defined in (3.15). For any \(k, l \in \mathbb{Z}, l \leq 0, k \geq l\), we define
\[
S^l_k = \{B(x, 2a^{-k}) \cap \overline{D_l(x_0)} : x \in N_k \cap D_l(x_0)\}.
\]
We define a graph with vertex set \(S^l = \bigsqcup_{k \in \mathbb{Z}, l \in \mathbb{Z}} S^l_k\), whose edges are the union of horizontal and vertical edges. In this case, the vertical edges are defined using predecessor relation as above and the horizontal edges are defined with respect to the space \(\overline{D_l(x_0)}\). That is \(B \cap \overline{D_l(x_0)}, B' \cap \overline{D_l(x_0)} \in S^l_k\) share a horizontal edge if and only if \(\lambda \cdot B \cap \lambda \cdot B' \cap \overline{D_l(x_0)} \neq \emptyset\). This graph with vertex set \(S^l\) can be viewed as a hyperbolic filling of the compact space \(\overline{D_l(x_0)}\).

### 3.4 Combinatorial description of the conformal gauge

The purpose of this section is to recall a combinatorial description of the conformal gauge essentially due to M. Carrasco Piaggo [Car13]. In this section, we fix a compact, doubling, uniformly perfect metric space \((X, d)\) and a hyperbolic filling \(\mathcal{S}_d = (\mathcal{S}, D_\mathcal{S})\) with horizontal parameter \(\lambda \geq 8\) and vertical parameter \(a > 1\) that satisfies (3.8).

Propositions 3.4 and 3.7 suggest the following strategy construct metrics that are in the conformal gauge of \((X, d)\). By changing the metric of the hyperbolic filling \(\mathcal{S}_d\) to another metric that is almost geodesic and bi-Lipschitz (in particular, quasi-isometric), every visual metric of its boundary is changed to a metric in the conformal gauge of \((X, d)\). Perhaps surprisingly, all metrics in the conformal gauge can be obtained in this manner (up to a bi-Lipschitz map) as explained in Theorem 3.14.

The change of metric in a hyperbolic filling is done using a **weight function** \(\rho : \mathcal{S} \to (0, 1)\) on its vertex set. We define
\[
\pi(B) = \prod_{B' \in g(B)} \rho(B'). \quad (3.25)
\]
A path \( \gamma = \{B_i\}_{i=1}^N \) in \( S_d \) is a sequence of vertices such that there is an edge between \( B_i \) and \( B_{i+1} \) for all \( i = 1, \ldots, N - 1 \). In this case, we say that \( \gamma \) is a path from \( B_1 \) to \( B_N \). A path is said to be simple, if no two vertices in the path are the same. A path is said to be horizontal (resp. vertical), if all the edges in the path are horizontal (resp. vertical). We define the \( \rho \)-length of a path \( \gamma = \{B_i\}_{i=1}^N \) by

\[
L_\rho(\gamma) = \sum_{i=1}^N \pi(B_i),
\]

where \( \pi \) is as defined in (3.25). For points \( x, y \in X \) and \( n \in \mathbb{N} \), then the set of paths \( \Gamma_n(x, y) \) is defined as

\[
\Gamma_n(x, y) = \left\{ \gamma = \{B_i\}_{i=1}^k : \gamma \text{ is a path from } B_1 \text{ to } B_k, x \in B_1, y \in B_k, B_1 \in S_n, B_k \in S_n \right\}.
\]

(3.27)

We remark that a path \( \gamma \in \Gamma_n(x, y) \) need not be a horizontal path.

For two distinct points \( x, y \in X \) and \( \alpha \geq 2 \), we define

\[
m_\alpha(x, y) = \max \{k : B \in S_k, x \in \alpha \cdot B, y \in \alpha \cdot B\},
\]

\[
c_\alpha(x, y) = \{B \in S_k : k = m_\alpha(x, y), x \in \alpha \cdot B, y \in \alpha \cdot B\},
\]

\[
\pi(c_\alpha(x, y)) = \max_{B \in c_\alpha(x, y)} \pi(B).
\]

(3.28)

**Assumption 3.11.** A weight function \( \rho : S \to (0, 1) \) may satisfy some of the following hypotheses:

(H1) (Quasi-isometry) There exist \( 0 < \eta_- \leq \eta_+ < 1 \) so that \( \eta_- \leq \rho(B) \leq \eta_+ \) for all \( B \in S \).

(H2) (Gromov product) There exists a constant \( K_0 \geq 1 \) such that for all \( B, B' \in S \) with \( B \sim B' \in S \), we have

\[
\pi(B) \leq K_0 \pi(B'),
\]

where \( \pi \) is as defined in (3.25).

(H3) (Visual parameter) There exists \( \alpha \in [2, \lambda/4] \) and a constant \( K_1 \geq 1 \) such that for any pair of points \( x, y \in X \), there exists \( n_0 \geq 1 \) such that if \( n \geq n_0 \) and \( \gamma \) is a path in \( \Gamma_n(x, y) \), then

\[
L_\rho(\gamma) \geq K_1^{-1} \pi(c_\alpha(x, y)),
\]

where \( \Gamma_n(x, y), L_\rho, \pi(c_\alpha(x, y)) \) are as defined in (3.27), (3.26), and (3.28) respectively.

The following observation concerns the stability of the above assumption under ‘finite perturbations’.

**Remark 3.12.** Let \( \rho, \rho' : S \to (0, 1) \) are two different weight functions such that the set \( \{B \in S : \rho(B) \neq \rho'(B)\} \) is finite. Then if \( \rho \) satisfies the hypotheses (H1), (H2), and (H3), then so does \( \rho' \) (with possibly different constants).
The weight function $\rho$ can be used to define a metric on $\mathcal{S}$ that is bi-Lipschitz equivalent to $D_{\mathcal{S}}$ as we recall below. We summarize the properties of the metric below.

**Lemma 3.13.** ([Car13, Lemma 2.3 and Proposition 2.4]) Let $(X, d)$ be a compact, doubling, uniformly perfect metric space with $\text{diam}(X, d) = \frac{1}{2}$, and let $\mathcal{S}_d = (\mathcal{S}, D_{\mathcal{S}})$ denote a hyperbolic filling with parameters $\lambda, \rho$ satisfying (3.11) and (3.8). Let $\rho : \mathcal{S} \to (0, 1)$ be a weight function that satisfies (H1) and (H2). Then there exists a metric $D_{\rho}$ on $\mathcal{S}$ such that:

(a) $D_{\rho}$ is bi-Lipschitz equivalent to $D_{\mathcal{S}}$; that is there exists $\Lambda \geq 1$ such that

$$\Lambda^{-1}D_{\mathcal{S}}(B, B') \leq D_{\rho}(B, B') \leq \Lambda D_{\mathcal{S}}(B, B'), \quad \text{for all } B, B' \in \mathcal{S};$$

(b) any simple vertical path $\gamma = \{B_i\}_{i=1}^n$ joining $B \in \mathcal{S}_m$ and $B' \in \mathcal{S}_{m'}$ satisfies

$$D_{\rho}(B, B') = \sum_{i=1}^{n-1} D_{\rho}(B_i, B_{i+1}) = \left| \log \frac{1}{\pi(B)} - \log \frac{1}{\pi(B')} \right|;$$

(c) $(\mathcal{S}, D_{\rho})$ is almost geodesic and Gromov hyperbolic.

(d) The identity map $\text{Id} : (\mathcal{S}, D_{\mathcal{S}}) \to (\mathcal{S}, D_{\rho})$ induces the identity map on their boundaries as described in Proposition 3.4. That is, a sequence $\{B_i\}$ converges at infinity in $(\mathcal{S}, D_{\mathcal{S}})$ if and only if it converges at infinity in $(\mathcal{S}, D_{\rho})$, and any two sequences that converge at infinity in $(\mathcal{S}, D_{\mathcal{S}})$ are equivalent if and only if they are equivalent in $(\mathcal{S}, D_{\rho})$. In particular, the bijection $p : X \to \partial(\mathcal{S}, D_{\mathcal{S}})$ described before Proposition 3.4 can be viewed as a bijection $\tilde{p} : X \to \partial(\mathcal{S}, D_{\rho})$ by composing with the induced identity map above.

(e) Assume in addition that (H3) is also satisfied. Let $(\cdot, \cdot)_\rho$ denote the Gromov product on $(\mathcal{S}, D_{\rho})$ with base point $w \in \mathcal{S}_0$ extended to its boundary. Define $\tilde{\theta}_\rho : \partial(\mathcal{S}, D_{\rho}) \times \partial(\mathcal{S}, D_{\rho}) \to [0, \infty)$ as

$$\tilde{\theta}_\rho(\tilde{p}(x), \tilde{p}(y)) = \inf \sum_{i=1}^{n-1} e^{-(\tilde{p}(x_i) + \tilde{p}(x_{i+1}))}, \quad (3.29)$$

where the infimum is over all finite sequence of points $\{x_i\}_{i=1}^n$ in $X$ such that $n \in \mathbb{N}$, $x_1 = x$, and $x_n = y$. Then $\tilde{\theta}_\rho$ is a visual metric on $\partial(\mathcal{S}, D_{\rho})$ with visual parameter $\epsilon$. Moreover, there exists $K > 1$ such that

$$K^{-1}e^{-(\tilde{p}(x) + \tilde{p}(y))} \leq \tilde{\theta}_\rho(\tilde{p}(x), \tilde{p}(y)) \leq Ke^{-(\tilde{p}(x) + \tilde{p}(y))},$$

$$K^{-1}\pi(c_\alpha(x, y)) \leq \tilde{\theta}_\rho(\tilde{p}(x), \tilde{p}(y)) \leq K\pi(c_\alpha(x, y)).$$

**Sketch of the proof.** We briefly recall the construction of the metric $D_{\rho}$. Let $E$ denote the edge set of the hyperbolic filling and let $\eta_-, \eta_+, K_0$ denote the constants in hypotheses (H1), and (H2). Define a function $\ell_\rho : E \to (0, \infty)$ as

$$\ell_\rho(e) = \begin{cases} 2 \max \{-\log(\eta_+), -\log(\eta_-), \log(K_0)\}, & \text{if } e \text{ is a horizontal edge,} \\ \log \frac{\pi(B')}{\pi(B)}, & \text{if } e = (B', B) \text{ is a vertical edge.} \end{cases}$$
The distance $D_\rho : \mathcal{S} \times \mathcal{S} \to [0, \infty)$ is defined as

$$D_\rho(B, B') = \inf_{\gamma} \sum_{i=1}^{N-1} \ell_{\rho}(e_i),$$

where the infimum is taken over all paths $\gamma = \{B_i\}_{i=1}^N$ where $N$ varies over $\mathbb{N}$, $B_1 = B, B_N = B'$ and $e_i = (B_i, B_{i+1})$ is an edge for each $i = 1, \ldots, N - 1$.

Part (a) is immediate from the definition of $D_\rho$. Part (b) and (c) are proved in [Car13, Lemma 2.3]. Part (d) follows from (a),(c) and Proposition 3.4. Part (e) follows from [Car13, Proposition 2.4]. □

The following theorem provides a combinatorial description of the conformal gauge $\mathcal{J}(X, d)$. In [Car13, Theorem 1.1] provides a combinatorial description of the Ahlfors regular conformal gauge $\mathcal{J}_{AR}(X, d) = \{ \theta \in \mathcal{J}(X, d) : \text{there exists an Ahlfors regular measure } \mu \text{ on } (X, \theta) \}$.

In [Car13, Theorem 1.1] the hypotheses (H1), (H2), (H3) corresponds to a combinatorial description of $\mathcal{J}(X, d)$, whereas the hypothesis (H4) corresponds to the existence of an Ahlfors regular measure. This theorem is essentially contained in [Car13].

**Theorem 3.14.** (Cf. [Car13, Theorem 1.1]) Let $(X, d)$ be a compact, doubling, uniformly perfect metric space.

(a) Let $\mathcal{S}_d = (\mathcal{S}, D_\mathcal{S})$ denote a hyperbolic filling with parameters $\lambda, a$ satisfying (3.11) and (3.8). Let $\rho : \mathcal{S} \to (0, 1)$ be a weight function that satisfies the conditions (H1), (H2), and (H3). Define the metric $\theta_\rho : X \times X \to [0, \infty)$ as

$$\theta_\rho(x, y) = \tilde{\theta}_\rho(\tilde{p}(x), \tilde{p}(y)) \quad \text{for } x, y \in X,$$

where $\tilde{\theta}_\rho$ is as defined in (3.29). Then $\theta_\rho$ satisfies the following properties:

(i) $\theta_\rho \in \mathcal{J}(X, d)$; that is $\theta_\rho$ is quasisymmetric to $d$.

(ii) there exists $C > 0$ such that

$$C^{-1} \pi(c_\alpha(x, y)) \leq \theta_\rho(x, y) \leq C \pi(c_\alpha(x, y)),$$

where $\alpha$ is the constant in (H3). Furthermore, there exists $K > 1$ such that

$$K^{-1} \pi(B) \leq \text{diam}(B, \theta_\rho) \leq K \pi(B) \quad \text{for all } B \in \mathcal{S}.$$

(iii) $\theta_\rho$ is a visual metric of the hyperbolic space $(\mathcal{S}, D_\rho)$ constructed in Lemma 3.13 in the following sense: there exists $C > 0$ such that

$$C^{-1} \theta_\rho(x, y) \leq e^{-(\tilde{p}(x)|\tilde{p}(y))_\rho} \leq C \theta_\rho(x, y),$$

where $\tilde{p} : X \to \partial(S, D_\rho)$ is the bijection described in Lemma 3.13(d), and $(\cdot | \cdot)_\rho$ denotes the Gromov product (extended to the boundary) on the hyperbolic space $(\mathcal{S}, D_\rho)$ with base point $w \in \mathcal{S}_0$. 28
The distortion function $\eta$ of the power quasisymmetry $\text{Id} : (X, d) \to (X, \theta)$ can be chosen to depend only on the constants in (H1), (H2), and (H3).

(b) Conversely, let $\theta \in \mathcal{J}(X, d)$ be any metric in the conformal gauge. Then there exists a hyperbolic filling $\mathcal{S}_d = (\mathcal{S}, D_\mathcal{S})$ of $(X, d)$ with horizontal parameter $\lambda$, vertical parameter $a$, and a weight function $\rho : \mathcal{S} \to (0, 1)$ that satisfies the hypotheses (H1), (H2), (H3), and such that the metric $\theta_\rho$ defined in (3.30) is bi-Lipschitz equivalent to $\theta$.

Proof. We begin with the proof of (b)

(b) Let $\text{Id} : (X, d) \to (X, \theta)$ be an $\eta$-quasisymmetry for some distortion function $\eta$.

The definition of the weight function $\rho$ in [Car13] uses an Ahlfors regular measure. Since there is no such measure available in our setting, the following definition is more suited for our purposes. We normalize the metric $\theta$, so that $\text{diam}(X, \theta) = \frac{1}{2}$. We will define the weight function $\rho : \mathcal{S} \to (0, 1)$ so that

$$\pi(B) = \text{diam}_\theta(B),$$

for all $B \in \mathcal{S}$, where $\mathcal{S}$ is a hyperbolic filling of $(X, d)$ with parameters $\lambda, a$. Fix any $\lambda \geq 32$. The vertical parameter $a > 1$ will be determined later in the proof. Hence we define $\rho : \mathcal{S} \to (0, 1)$ as

$$\rho(B) = \begin{cases} \frac{1}{2} \frac{\text{diam}_\theta(B)}{\text{diam}_\theta(B_{\lambda - 1})} & \text{if } B \in \mathcal{S}_0, \\ \frac{\text{diam}_\theta(B)}{\text{diam}_\theta(B_{\lambda - 1})} & \text{if } B \in \mathcal{S}_n, n \geq 1. \end{cases}$$

First, we show (H2). Let $B \sim B'$ with $B, B' \in \mathcal{S}_n$. Then choose $y \in \lambda \cdot B \cap \lambda \cdot B'$. By triangle inequality,

$$B \subset B_d(x_B, 2a^{-n}) \subset B_d(y, (\lambda + 2)a^{-n}), \quad B' \subset B_d(y, (\lambda + 2)a^{-n}).$$

By uniform perfectness, and triangle inequality, for any $r < \frac{1}{2}, r/K_P \leq \text{diam}_\theta(B_d(x, r)) \leq 2r$. Therefore by (3.5), we obtain

$$\frac{1}{2\eta(4(\lambda + 2)K_P)} \leq \frac{\text{diam}_\theta(B)}{\text{diam}_\theta(B_d(y, (\lambda + 2)a^{-n}))} \leq \eta(8K_P/(\lambda + 2)).$$

Since the same inequality holds with $B$ replaced with $B'$, we have (H2) with constant

$$K_0 = 2\eta(4(\lambda + 2)K_P)\eta(8K_P/(\lambda + 2)),$$

that depends only on the distortion function $\eta$, the constant $K_P$ of uniform perfectness, and the horizontal parameter $\lambda$ (in particular, does not depend on the vertical parameter $a$).

Next, we show (H1), which again relies on (3.5). We will choose $a > 2(\lambda + 1)$ large enough so that $\eta_+ = \frac{1}{2}$ in (H1). Clearly this choice works when $B \in \mathcal{S}_0$. If $B = \mathcal{S}_n, n \geq 1,$
and by denoting $B' = g(B)_{n-1}$, we have $x_B \in B'$. For $n \geq 2$, we write (the case $n = 1$ is easier and left to the reader)

$$\rho(B) = \frac{\text{diam}_\theta(B)}{\text{diam}_\theta(B')} = \frac{\text{diam}_\theta(B)}{\text{diam}_\theta((4a) \cdot B)} \frac{\text{diam}_\theta((4a) \cdot B)}{\text{diam}_\theta(B')}.$$  

Each of the terms can be estimated (from above and below) using (3.5), since by the triangle inequality and $d(x_B, x_{B'}) < 2a^{-n+1}$ we have $B \subseteq (4a) \cdot B$, and $(4a) \cdot B \supseteq B(x_B, 2a^{-n-1}) \supseteq B'$. Hence, we obtain

$$\rho(B) \leq 2\eta \left( \frac{2\text{diam}_d(B)}{\text{diam}_d((4a) \cdot B)} \right) \eta \left( \frac{\text{diam}_d((4a) \cdot B)}{\text{diam}_d(B')} \right) \leq 2\eta (K_P/a) \eta (16K_P)$$  

$$\rho(B) \geq \left[ 2\eta \left( \frac{\text{diam}_d((4a) \cdot B)}{\text{diam}_d(B)} \right) \eta \left( \frac{\text{diam}_d(B)}{\text{diam}_d((4a) \cdot B)} \right) \right]^{-1} \geq [2\eta (8K_P) \eta (2K_P/a)]^{-1}$$

First we choose $a$ large enough so that $2\eta (K_P/a) \eta (16K_P) \leq \frac{1}{2}$ and (3.11) are satisfied. We set $\eta_\gamma = [2\eta (8K_P) \eta (2K_P/a)]^{-1}$. Hence we obtain (H1).

For (H3), we once again use (3.5), to see that $\pi(B) = \text{diam}_\theta(B)$ is comparable to $\text{diam}_\theta(\lambda \cdot B)$ for all $B \in \mathcal{S}$. More precisely, we have

$$\text{diam}_\theta(B) \leq \text{diam}_\theta(\lambda \cdot B) \leq 2\eta (2\lambda K_P) \text{diam}_\theta(B)$$

for all $B \in \mathcal{S}$. For any path $\gamma = \{B_i\}_{i=1}^m \in \Gamma_n(x, y)$, we choose points $x_i \in \lambda \cdot B_i \cap \lambda \cdot B_{i+1}, i = 1, \ldots, m-1$, $x_0 = x, x_m = y$ so that

$$\theta(x, y) \leq \sum_{i=0}^{m-1} \theta(x_i, x_{i+1}) \leq \sum_{i=1}^m \text{diam}_\theta(\lambda \cdot B_i) \leq \sum_{i=1}^m \text{diam}_\theta(B_i) = 2\eta (2\lambda K_P) L_\rho(\gamma). \quad (3.33)$$

Fixing $\alpha = 2$, and let $C \in c_2(x, y)$ such that $\pi(c_2(x, y)) = \pi(C)$. Let $m = m_2(x, y)$. Let $B \in \mathcal{S}_{m+1}$ be such that $x \in B$. By definition of $m_2(x, y)$, $y \not\in 2 \cdot B$. Therefore $d(x, y) \geq d(x_B, y) - d(x_B, x) \geq a^{-m-1}$. By (3.5), and $\pi(c_2(x, y)) \leq \text{diam}_\theta(2 \cdot C)$ we have

$$\pi(c_2(x, y)) \leq 2\eta \left( \frac{\text{diam}_d(2 \cdot C)}{d(x, y)} \right) \theta(x, y) \leq 2\eta \left( \frac{8a^{-m}}{a^{-m-1}} \right) \theta(x, y) = 2\eta (8a) \theta(x, y) \quad (3.34)$$

Combining (3.33) and (3.34) yields (H3) with $\alpha = 2$.

(a) This part is essentially contained in [Car13]. The hypotheses (H1) and (H2) are used to construct a metric $D_\rho$ on $\mathcal{S}$ as given in Lemma 3.13. If $\theta_\rho$ were defined using (3.30), it clearly satisfies the symmetry $\theta_\rho(y, x) = \theta_\rho(x, y)$, and triangle inequality. The role of (H3) is to show that $\theta_\rho(x, y)$ is at least $e^{-\bar{\rho}(x)\bar{\rho}(y))}$ (up to a constant factor) as explained in Lemma 3.13(e). The fact that $\theta_\rho$ is quasisymmetric to $d$ follows from Lemma 3.13, Propositions 3.7 and 3.4(c). The statement about the dependence of distortion function $\eta$ on the constants follow from Remark 3.5.
The estimate (3.32) is also implicitly contained in [Car13] and is a consequence of (3.31). Choose \( x, y \in B \) such that \( d(x, y) \geq \text{diam}(B, d)/2 \). Since \( \text{Id} : (X, d) \rightarrow (X, \theta_\rho) \) is an \( \eta \)-quasisymmetry, by (3.5), there exists \( C_1 > 0 \) such that

\[
\theta_\rho(x, y) \leq \text{diam}(B, \theta_\rho) \leq C_1 \theta_\rho(x, y).
\]

Since \( d(x, y) \geq \text{diam}(B)/2 \), \( B \) is at a bounded distance in \((S, D_S)\) from any set \( C \in c_\alpha(x, y) \). Combining these estimates along with (3.31) and (3.5), we obtain

\[
\text{diam}(B, \theta_\rho) \preceq \theta_\rho(x, y) \preceq \pi(c_\alpha(x, y)).
\]

\[\Box\]

### 3.5 Construction of measure using the hyperbolic filling

As in §3.2, we fix a compact, doubling, uniformly perfect metric space \((X, d)\) with \( \text{diam}(X, d) = \frac{1}{2} \), and a hyperbolic filling \( S_d = (S, D_S) \) with horizontal parameter \( \lambda \) and vertical parameter \( a \) that satisfy (3.11).

**Definition 3.15** (gentle function). Let \( C : S \rightarrow (0, \infty) \) and \( K \geq 1 \). We say that \( C \) is \( K \)-gentle if

\[
K^{-1} C(B') \leq C(B) \leq KC(B),
\]

whenever there is an edge between \( B \) and \( B' \). We say that \( C : S \rightarrow (0, \infty) \) is gentle if it is \( K \)-gentle for some \( K \geq 1 \). The notion of \( K \)-gentle function extends to any function \( f : V \rightarrow (0, \infty) \) on a graph \( G = (V, E) \). In other words, a function \( f : V \rightarrow (0, \infty) \) on is \( K \)-gentle if \( \log f \) is \((\log K)\)-Lipschitz with respect to the graph distance metric.

We sometimes need to distinguish between the horizontal and vertical edges (see Theorem 3.24). We say \( C : S \rightarrow (0, \infty) \) is \((K_h, K_v)\)-gentle if

\[
K_h^{-1} C(B') \leq C(B) \leq K_h C(B'),
\]

whenever \( B \) and \( B' \) share a horizontal edge, and

\[
K_v^{-1} C(B') \leq C(B) \leq K_v C(B'),
\]

whenever \( B \) and \( B' \) share a vertical edge. Therefore every \((K_h, K_v)\)-gentle function is \((K_h \lor K_v)\)-gentle.

Given a hyperbolic filling \( S \), we need to approximate a ball \( B(x, r) \) by a ball in the filling \( S \). We introduce this notion in the following definition.

**Definition 3.16.** Let \((X, d)\) be a doubling metric space. Let \((S, D_S)\) be a hyperbolic filling of \((X, d)\) with parameters \( a, \lambda \) that satisfy (3.11) as constructed in Lemma 3.9(a). By Lemma 3.9(a), given a ball \( B(x, r) \neq X \), there exists \( n \in \mathbb{Z} \) and \( B \in S_n \) such that

\[
2a^{-n-1} \leq r < 2a^{-n}, \quad \text{and} \quad d(x_B, x) < 2a^{-n}.
\]
We define

\[ A_S(B(x, r)) = \{ B \in S : n \in \mathbb{Z}_{\geq 0} \text{ and } B \in S_n \text{ satisfy (3.35)} \} . \] (3.36)

We remark that \( B, B' \in A_S(B(x, r)) \), then \( x \in B \cap B' \neq \emptyset \) and hence \( B \) and \( B' \) share a horizontal edge.

Often, the measures in this work will satisfy the following volume doubling and reverse volume doubling properties.

**Definition 3.17** (Volume doubling and Reverse volume doubling properties). We say that a non-zero Radon measure \( \mu \) on a metric space \((X, d)\) satisfies the volume doubling property VD, if there exists \( C_D \in (1, \infty) \) such that

\[ \mu(B(x, 2r)) \leq C_D \mu(B(x, r)), \quad \text{for all } x \in X, r \in (0, \infty). \] (VD)

A non-zero Radon measure \( \mu \) on a metric space \((X, d)\) satisfies the reverse volume doubling property RVD, if there exist \( C_1, C_2 \in (1, \infty), \alpha \in (0, \infty) \) such that

\[ \mu(B(x, R)) \geq C_1^{-1} \left( \frac{R}{r} \right)^\alpha \mu(B(x, r)), \quad \text{for all } 0 < r \leq R < \text{diam}(X, d)/C_2. \] (RVD)

**Remark 3.18.** We recall the following connection between the doubling and uniform perfectness properties on a metric space \((X, d)\) to the volume doubling and reverse volume doubling properties.

(a) If \( \mu \) satisfies VD on \((X, d)\), then \((X, d)\) is a doubling metric space. Conversely, every complete doubling metric space admits a measure that satisfies VD [Hei, Theorem 13.3]. The constant 2 in the definition of VD is essentially arbitrary, as VD implies

\[ \frac{\mu(B(x, R))}{\mu(B(x, r))} \leq C_D \left( \frac{R}{r} \right)^\alpha, \quad \text{for all } x \in X, 0 < r \leq R, \text{ where } \alpha = \log_2 C_D. \] (3.37)

(b) Let \( \mu \) be a measure that satisfies VD on \((X, d)\). Then \( \mu \) satisfies RVD if and only if \((X, d)\) is uniformly perfect [Hei, Exercise 13.1].

We introduce a hypothesis on a weight function \( \rho : S \to (0, \infty) \) that plays an important role in the construction of a measure.

**Assumption 3.19.** Let \( \mathcal{C} : S \to (0, \infty) \) be a gentle function, and let \( \beta > 0 \). A weight function \( \rho : S \to (0, 1) \) is said to be \((\beta, \mathcal{C})\)-compatible if there exists \( K_2 \geq 1 \) such that for all \( B \in S_m, \) and \( n > m, \)

\[ K_2^{-1} \pi(B)^\beta \mathcal{C}(B) \leq \sum_{B' \in \mathcal{D}_n(B)} \pi(B')^\beta \mathcal{C}(B') \leq K_2 \pi(B)^\beta \mathcal{C}(B), \]

where \( \mathcal{D}_n(B) \) denotes the descendants of \( B \) of generation \( n \) as defined in (3.10).
The above assumption is similar to (H4) in [Car13]. The following lemma is an analogue of [Car13, Lemma 2.7].

**Lemma 3.20.** Let \((S, D_S)\) be a hyperbolic filling of a doubling, \(K_p\)-uniformly perfect, compact metric space as given in Lemma 3.9(a). Let \(\rho : S \rightarrow (0, 1)\) be a weight function that satisfies (H1) and (H2). Let \(C : S \rightarrow (0, \infty)\) be a gentle function, and let \(\beta > 0\), such that \(\rho\) is \((\beta, S)\)-compatible. For \(n \geq 0\), denote

\[
\mu_n := \sum_{B \in S_n} \pi(B)^{\beta} C(B) \delta_{x_B},
\]

where \(\delta_{x_B}\) denotes the Dirac measure at \(x_B\). Let \(\mu\) be any weak* subsequential limit of \(\mu_n\). Then there exists \(C_1 > 1\) such that, for all \(x \in X, r \leq \text{diam}(X)/2\), and for all \(B \in A_S(B(x,r))\), we have

\[
C_1^{-1} \pi(B)^{\beta} C(B) \leq \mu((B(x,r))) \leq C_1 \pi(B)^{\beta} C(B),
\]

(3.38)

where \(A_S\) is as given in Definition 3.16. Furthermore, \(\mu\) is satisfies VD on \((X,d)\).

**Sketch of the proof.** We only sketch the proof and skip the details as it follows from almost the same argument as [Car13, Lemma 2.7].

Let \(x \in X, r \leq \text{diam}(X)/2, B \in A_S(B(x,r))\) and \(B = B(x_B, 2a^{-m})\). Choose \(B_1 \in S_{m+2}\) such that \(x \in B_1 = B(x_B, 2a^{-m-2})\). By [Car13, (2.10)], the centers of all the descendants of \(B_1\) belong to \(B(x,r/2)\). This along with \((\beta, C)\)-compatibility implies that

\[
\mu(B(x,r)) \geq \mu(B(x,r/2))
\]

\[
\geq \liminf_{n \to \infty} \mu_n (\{x_C : C \in D_n(B_1)\})
\]

\[
\geq \liminf_{n \to \infty} \sum_{B' \in D_n(B_1)} \pi(B')^{\beta} C(B')
\]

\[
\geq \pi(B_1)^{\beta} C(B_1) \geq \pi(B)^{\beta} C(B) \quad \text{(by \((\beta, C)\)-compatibility and gentleness of } C\).
\]

By the argument in [Car13, proof of Lemma 2.7], for any \(B' \in S_n, n \geq m\) satisfying \(x_{B'} \in B(x,r + a^{-m})\), we have \(g(B')_n \sim g(B_1)_m\), where \(B_1\) is as defined above. For the upper bound, for any \(B' \in S_n, n \geq m\) such that \(x_{B'} \in B(x,r + a^{-m})\), we have that \(g(B')_m \sim B\). Therefore, we estimate

\[
\mu(B(x,r)) \leq \mu(B(x,r + a^{-m}))
\]

\[
\leq \limsup_{n \to \infty} \mu_n(B(x,r + a^{-m}))
\]

\[
\leq \sum_{C \sim g(B_1)_m} \sum_{B' \in D_n(C)} \pi(B')^{\beta} C(B')
\]

\[
\lesssim \sum_{C \sim g(B_1)_m} \pi(C)^{\beta} C(C).
\]

Since \(C \sim g(B_1)_m\) and \(B \sim g(B_1)_m\), by gentleness of \(C\), we have \(C(C) \approx C(B)\). By (H2), we have \(\pi(C) \asymp \pi(B)\). Furthermore, by doubling the number of such \(C \in S_m\) such that
$C \sim g(B_1)_m$ is bounded by a constant that depends only on the parameters of the filling. Combining these estimates, we obtain the desired upper bound $\mu(B(x, r)) \lesssim \pi(B)^\beta C(B)$. This completes the proof of (3.38).

The conclusion that $\mu$ satisfies VD follows from (3.38) and the gentleness of $C$. □

In the following proposition, we express the measure in Lemma 3.20 using the metric in Theorem 3.14(a).

**Proposition 3.21.** Let $(X, d)$ be a compact, doubling, uniformly perfect metric space. Let $(\mathcal{S}, D_\mathcal{S})$ be a hyperbolic filling with parameters $\lambda, \nu$ satisfying (3.11), (3.8) as given in Lemma 3.20. Let $C : \mathcal{S} \to (0, \infty)$ be a gentle function and let $\beta > 0$. Let $\rho : C \to (0, 1)$ be a weight function that satisfies (H1), (H2), (H3), and $(\beta, C)$-compatibility. Let $\theta = \theta_\rho \in \mathcal{J}(X, d)$ denote the metric in Theorem 3.14(a) and $\mu$ denote the measure on $X$ constructed in Lemma 3.20. Then, there exist $C_1 > 1$ such that

$$C_1^{-1}r^\beta C(B) \leq \mu(B_\theta(x, r)) \leq C_1r^\beta C(B), \quad \text{for all } x \in X, r < \text{diam}(X, \theta), B \in A_S(B_d(x, s)),$$

(3.39)

where $s$ is the largest number in $[0, 2\text{diam}(X, d)]$ such that $B_d(x, s) \subset B_\theta(x, r)$ (as defined in (3.2)) and $A_S(B_d(x, s))$ is as given in Definition 3.16.

**Proof.** By an easy covering argument using the metric doubling property, it suffices to consider the case $r < \text{diam}(X, \theta)/2$, so that $B_\theta(x, r) \neq X$.

Let $x \in X, 0 < r < \text{diam}(X, \theta)/2$ and let $s = \sup s_1 > 0 : B_d(x, s_1) \subset B_\theta(x, r)$. By Lemma 3.20, $\mu$ satisfies VD in $(X, d)$. By (3.1) and in (3.37), $\mu$ satisfies VD in $(X, \theta)$ and there exists $C_2 > 1$ such that

$$C_2^{-1}\mu(B_d(x, s)) \leq \mu(B_\theta(x, r)) \leq C_2\mu(B_d(x, s)).$$

(3.40)

By (3.1), there exists $A_1 > 1$ such that $B_\theta(x, r) \subset A_1 \cdot B$ and $B \subset B_\theta(x, A_1 r)$ for all $B \in A_S(B_d(x, s))$. Hence by (3.5) and uniform perfectness, there exists $C_3 > 1$ such that

$$C_3^{-1}r \leq \text{diam}(B, \theta) \leq C_3r, \quad \text{for all } B \in A_S(B_d(x, s)).$$

(3.41)

By (3.32), (3.40), and (3.41), we obtain (3.39). □

### 3.6 Simplified hypotheses for construction of metric and measure

The goal of this section is to present an analogue of [Car13, Theorem 1.2] that will be used in the construction of metric measure space. Some of the main ideas in the proof of [Car13, Theorem 1.2] are inspired by the ‘weight-loss program’ of Keith and Laakso [KL, §5.2].

We continue to consider a compact, doubling, uniformly perfect metric space $(X, d)$, and a hyperbolic filling $\mathcal{S}_d = (\mathcal{S}, D_\mathcal{S})$ with horizontal parameter $\lambda \geq 8$ and vertical
parameter $a > 1$ that satisfy (3.8). We consider $\beta > 0$, $\mathcal{C} : \mathcal{S} \rightarrow (0, \infty)$ such that $\mathcal{C}$ is gentle. Theorem 3.24 provides simpler sufficient conditions (S1), (S2) that allows us to construct a weight function that satisfies (H1), (H2), (H3), and is $(\beta, \mathcal{C})$-compatible. To state the sufficient conditions, we recall the following definition.

**Definition 3.22.** For $B \in \mathcal{S}_k, k \geq 0$, we define $\Gamma_{k+1}(B)$ as the set of horizontal paths $\gamma = \{B_i\}_{i=1}^N, N \geq 2$ such that $B_i \in \mathcal{S}_{k+1}$ for all $i = 1, \ldots, N$, $B_i \sim B_{i+1}$ for all $i = 1, \ldots, N - 1$, $x_{B_i} \in B$, $x_{B_N} \notin 2 \cdot B$, and $x_{B_i} \in 2 \cdot B$ for all $i = 1, \ldots, N - 1$.

We introduce a subadditive estimate based on [BM, Proposition 3.15].

**Definition 3.23.** We say that $B \in \mathcal{S}_k, k \geq 1$ is non-peripheral if every horizontal neighbour of $B$ descends from the same parent. More precisely, $B \in \mathcal{S}_k, k \geq 1$ is non-peripheral if

$$B \sim B' \text{ implies that } g(B)_{k-1} = g(B')_{k-1}. $$

By $\mathcal{N}$ we denote the set of all non-peripheral vertices in $\mathcal{S}$. We say that a function $\mathcal{C} : \mathcal{S} \rightarrow (0, \infty)$ satisfies (E) if it obeys the following estimate:

$$(E) \text{ there exists } \delta \in (0, 1) \text{ such that } \mathcal{C}(B) \leq (1 - \delta) \sum_{B' \in \mathcal{N} \cap \mathcal{D}_{k+1}(B)} \mathcal{C}(B')$$

for all $B \in \mathcal{S}_k, k \geq 1$.

In particular, the condition (E) implies $\mathcal{N} \cap \mathcal{D}_{k+1}(B) \neq \emptyset$ for all $k \geq 1, B \in \mathcal{S}_k$.

The following result is an analogue of [Car13, Theorem 1.2].

**Theorem 3.24.** Let $(X, d)$ be a compact, $K_D$-doubling, $K_P$-uniformly perfect metric space and let $\beta > 0$. Consider a hyperbolic filling $\mathcal{S}_d = (\mathcal{S}, D_{\mathcal{S}})$ with horizontal parameter $\lambda \geq 8$ and vertical parameter $a > 1$ that satisfies (3.8). Let $\mathcal{C} : \mathcal{S} \rightarrow (0, \infty)$ be a $(K_h, K_v)$-gentle function that satisfies (E). Then, there exists $\eta_0 \in (0, 1)$ that depends only on $\beta, K_D, K_h, \lambda$ (but not on the vertical constants $a, K_v$ or uniform perfectness constant $K_P$) such that the following is true. If there exists a function $\sigma : \mathcal{S} \rightarrow [0, \frac{1}{4})$ that satisfies:

(S1) for all $B \in \mathcal{S}_k, k \geq 0$, if $\gamma = \{B_i : 1 \leq i \leq N\}$ is a path in $\Gamma_{k+1}(B)$ (as given in Definition 3.22), then

$$\sum_{i=1}^N \sigma(B_i) \geq 1,$$

(S2) and for all $k \geq 0$, and all $B \in \mathcal{S}_k$, we have

$$\sum_{B' \in \mathcal{D}_{k+1}(B)} \sigma(B')^{\beta} \mathcal{C}(B') \leq \eta_0 \mathcal{C}(B),$$

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then there exists a weight function $\rho : \mathcal{S} \to (0,1)$ that satisfies (H1), (H2), (H3), and is $(\beta, \mathcal{C})$-compatible.

We recall some results from [Car13] that goes into the proof of Theorem 3.24.
Let $\rho : \mathcal{S} \to (0, \infty)$ be a function, we define $\rho^* : \mathcal{S} \to (0, \infty)$ as
$$\rho^*(B) = \min_{B' \sim B} \rho(B'), \quad \text{for } B \in \mathcal{S}.$$ We recall that $B \sim B'$ if there exists $k \geq 0$ such that $B, B' \in \mathcal{S}_k$ and $(\lambda \cdot B) \cap (\lambda \cdot B') \neq \emptyset$.

If $\gamma = \{B_i\}_{i=1}^{N}$ is a horizontal path, we define
$$L_h(\gamma, \rho) = \sum_{j=1}^{N-1} \rho^*(B_j) \wedge \rho^*(B_{j+1}).$$

**Proposition 3.25.** ([Car13, Proposition 2.9]) Let $(X, d)$ be a compact, doubling, and uniformly perfect metric space. Let $\mathcal{S}$ be a hyperbolic filling with parameters $a$ and $\lambda$ satisfying (3.11). Assume that $\rho : \mathcal{S} \to (0,1)$ satisfies (H1), (H2), and also the condition (H3') for all $k \geq 1$, for all $B \in \mathcal{S}_k$ and for all $\gamma \in \Gamma_{k+1}(B)$, it holds $L_h(\gamma, \rho) \geq 1$.

Then the function $\rho$ also satisfies (H3).

[Car13, Propostion 2.9] also assumes an additional assumption (H4) which was not used in the proof. In [Car13], the condition (H3') was stated for $k \geq 0$ but it is equivalent to the above condition because $\Gamma_1(B) = \emptyset$ for $B \in \mathcal{S}_0$.

**Lemma 3.26.** ([Car13, Lemma 2.13]) Suppose we have a function $\pi_0 : \mathcal{S}_k \to (0, \infty)$ such that
$$\forall B \sim B' \in \mathcal{S}_k, \quad \frac{1}{K} \leq \frac{\pi_0(B)}{\pi_0(B')} \leq K,$$
where $K \geq 1$ is a constant. Suppose that we have a function $\pi_1 : \mathcal{S}_{k+1} \to (0, \infty)$ which satisfies the following property:
$$\forall B \in \mathcal{S}_{k+1}, \exists A \in \mathcal{S}_k \text{ with } d(x_B, x_A) \leq 4a^{-k} \text{ and } 1 \leq \frac{\pi_0(A)}{\pi_1(B)} \leq K.$$ Define $\hat{\pi}_1 : \mathcal{S}_{k+1} \to (0, \infty)$ as
$$\hat{\pi}_1(B') = \pi_1(B') \lor \left( \frac{1}{K} \max \{\pi_1(B) : B \sim B'\} \right).$$ Then, for all $B \sim B' \in \mathcal{S}_{k+1}$, we have
$$\frac{1}{K} \leq \frac{\hat{\pi}_1(B)}{\hat{\pi}_1(B')} \leq K.$$ The following is a slight modification of [Car13, Lemma 2.14].
Lemma 3.27. (compare with [Car13, Lemma 2.14]) Let \( G = (V, E) \) be a graph whose vertices have degree bounded by \( D \). Let \( C : V \to (0, \infty) \) be \( K \)-gentle; that is,
\[
K^{-1} \leq \frac{C(z)}{C(z')} \leq K, \text{ whenever there is an edge between } z \text{ and } z'.
\]
Let \( \Gamma \) be a family of paths in \( G \) and let \( \beta > 0 \). Suppose that \( \tau : V \to (0, \infty) \) is a function satisfying
\[
\sum_{i=1}^{N-1} \tau(z_i) \geq 1, \text{ for all paths } \gamma = \{z_i\}_{i=1}^{N} \in \Gamma. \tag{3.42}
\]
Define \( \tilde{\tau} : V \to (0, \infty) \) as
\[
\tilde{\tau}(x) = 2 \max \{\tau(y) : y \in V_2(x)\},
\]
where \( V_2(x) \) denotes the set of all vertices whose graph distance from \( x \) is less than or equal to 2. Then \( \tilde{\tau} \) satisfies
\[
\sum_{i=1}^{N-1} \tilde{\tau}^*(z_i) \wedge \tilde{\tau}^*(z_{i+1}) \geq 1 \text{ for all paths } \gamma = \{z_i\}_{i=1}^{N} \in \Gamma,
\]
where \( \tilde{\tau}^*(x) = \min \{\tilde{\tau}(y) : y \sim x\} \), and such that
\[
\sum_{z \in V} \tilde{\tau}(z) \beta C(z) \leq 2^\beta D^2 K^2 \sum_{z \in V} \tau(z) \beta C(z). \tag{3.43}
\]

Proof. As shown in [Car13, Lemma 2.14] the function \( \tilde{\tau} \) satisfies (3.42).

Since \( C \) is \( K \)-gentle and \( \sup_{x \in V} |V_2(x)| \leq D^2 \), we obtain
\[
\sum_{x \in V} \tilde{\tau}(x) \beta C(x) \leq 2^\beta \sum_{x \in V} \sum_{z \in V_2(x)} \tau(z) \beta C(x)
\leq 2^\beta D^2 \sum_{x \in V} \sum_{z \in V_2(x)} \tau(z) \beta C(z) = 2^\beta \sum_{z \in X} \sum_{x \in V_2(z)} \tau(z) \beta C(z)
\leq 2^\beta K^2 D^2 \sum_{z \in X} \tau(z) \beta C(z).
\]

Proof of Theorem 3.24. Let \( D_h \) be such that
\[
D_h \geq \sup_{k \geq 0} \max_{B \in S_k} |\{B' \in S_{k'} : B' \sim B\}|. \tag{3.44}
\]

By \( K_D \)-doubling, \( D_h \) and can be chosen to depend only on \( \lambda \) and \( K_D \) [Hei, Exercise 10.17]. Similarly, the number of children can be bounded by a constant \( D_v \) that depends only on \( a \) and \( K_D \) with
\[
D_v \geq \sup_{k \geq 0} \max_{B \in S_k} |D_{k+1}(B)|. \tag{3.45}
\]
Take $\eta_0 \in (0, 1)$ be a constant which will be fixed later, and set
\[
\eta_- := (\eta_0 K_v^{-1} D_v^{-1})^{1/\beta} \wedge \frac{1}{4}.
\]

Let $\sigma : S \to [0, \frac{1}{4})$ satisfy (S1) and (S2). Define $\tau = \sigma \lor \eta_-$. Then
\[
\sum_{B' \in D_{k+1}(B)} \tau(B') \beta C(B') \leq \sum_{B' \in D_{k+1}(B)} \sigma(B') \beta C(B') + \eta_0^\beta D_v K_v C(B) \leq 2\eta_0 C(B).
\]

For $B \in S_k$ define $V_{2,k}(B) = \{B' \in S_k : \exists B'' \in S_k \text{ such that } B \sim B'' \sim B\}$. The by Lemma 3.27, the function
\[
\tilde{\tau}(B) = 2 \max \{\tau(B') : B' \in V_{2,k}(B)\}, \quad \text{for all } B \in S_k
\]
satisfies (H3') and
\[
\sum_{B' \in D_{k+1}(B)} \tilde{\tau}(B') \beta C(B') \leq 2^\beta K_h^2 D_h^2 \sum_{B' \in D_{k+1}(B)} \tau(B') \beta C(B') \leq 2^\beta K_h^2 D_h^2 \eta_0 C(B), \quad (3.46)
\]
for all $B \in S_k$.

We construct $\hat{\rho} : S \to (0, 1)$ satisfying

(1) $\hat{\rho} \geq \tilde{\tau}$. In particular $\hat{\rho}$ satisfies (H3') and $\hat{\rho}(B) \geq \eta_-$ for all $B \in S$.

(2) (H2) with constant $K$, where $K = \eta_-^{-1}$.

(3) $\hat{\rho}(B) \leq \max \{\tilde{\tau}(B') : B' \sim B\}$. In particular, $\hat{\rho}(B) \leq \frac{1}{2}$ for all $B \in S$.

We briefly recall the construction in [Car13]. Set $\hat{\rho}(w) = \frac{1}{2}$, where $w \in S_0$. Note that $\tilde{\tau} \leq \frac{1}{2} \leq 1$ (since $\eta_- \leq \frac{1}{2}$ and $\sigma \leq \frac{1}{4}$). We construct $\hat{\rho}$ inductively on $S_k$. Suppose we have constructed $\hat{\rho}_i$ for $i = 1, \ldots, k$. We construct $\hat{\rho}_{k+1}$ using Lemma 3.26. We denote
\[
\pi_0(A) = \prod_{i=0}^{k} \hat{\rho}_i(g(A)_i) \text{ for } A \in S_k \text{ and, } \pi_1(B) = \tilde{\tau}(B) \pi_0(g(B)_j) \text{ for } B \in S_{k+1}.
\]

By the induction hypothesis along with Lemma 3.26, we obtain a function $\hat{\pi}_1 : S_{j+1} \to (0, \infty)$ that satisfies $K^{-1} \hat{\pi}_1(B') \leq \hat{\pi}_1(B) \leq K \hat{\pi}_1(B')$ for all $B \sim B' \in S_{j+1}$. We define $\hat{\rho} : S_{j+1} \to (0, \infty)$ as
\[
\hat{\rho}_{k+1}(B) = \frac{\hat{\pi}_1(B)}{\pi_0(g(B)_j)}.
\]

Carrasco Piaggo’s proof of [Car13, Theorem 1.2] shows that $\hat{\rho}$ satisfies properties (1), (2),
and (3) above. For any $B \in S_k$, $k \geq 0$, using (3.46) we estimate

$$\sum_{B' \in D_{k+1}} \hat{\rho}(B')^\beta C(B')$$

$$\leq \sum_{B' \in D_{k+1}(B)} \sum_{B'' \sim B'} \tilde{\tau}(B'')^\beta C(B'') \quad \text{(by property (3) above)}$$

$$\leq K_h \sum_{B' \in D_{k+1}(B)} \sum_{B'' \sim B'} \tilde{\tau}(B'')^\beta C(B'') \quad (\because B'' \sim B' \implies g(B'')_k \sim B)$$

$$\leq 2^{\beta+1} K_h^3 D_h^3 \eta_0 \sum_{C \sim B} C(C) \leq 2^{\beta+1} K_h^4 D_h^4 \eta_0 C(B).$$

(3.47)

Now choose $\eta_0$

$$2^{\beta+1} K_h^3 D_h^3 \eta_0 = \frac{1}{2},$$

(3.48)

so that (3.47) yields

$$\sum_{B' \in D_{k+1}} \hat{\rho}(B')^\beta C(B') \leq \frac{1}{2} C(B) \quad \text{for all } B \in S_k, k \geq 0.$$

(3.49)

Note from (3.48) that $\eta_0$ depends only on $\beta, K_h, K_D, \lambda$ but not on constants $K_v, a, K_P$.

Next, we modify $\hat{\rho}$ so that it becomes $(\beta, C)$-compatible. For each $B \in S_k, k \geq 0$, we choose $\omega_B \geq 0$ such that

$$\rho(B') = \begin{cases} 
\omega_B \lor \hat{\rho}(B') & \text{if } B' \in D_{k+1}(B) \cap \mathcal{N}, \\
\hat{\rho}(B') & \text{if } B' \in D_{k+1}(B) \setminus \mathcal{N}.
\end{cases}$$

satisfies

$$\sum_{B' \in D_{k+1}(B)} \rho(B')^\beta C(B') = C(B).$$

(3.50)

The existence of an $\omega_B \in (0, \infty)$ that satisfies (3.50) follows from the intermediate value theorem. In particular, we use (3.49), the continuity of the map

$$\omega_B \mapsto \sum_{B' \in D_{k+1} \cap \mathcal{N}} (\omega_B \lor \rho(B'))^\beta C(B') + \sum_{B' \in D_{k+1} \setminus \mathcal{N}} \hat{\rho}(B')^\beta C(B'),$$

along with the fact that $D_{k+1} \cap \mathcal{N}$ is non-empty. The equality (3.50) implies that $\rho$ is $(\beta, C)$-compatible since

$$\sum_{B' \in D_n(B)} \pi(B')^\beta C(B') = \pi(B)^\beta C(B)$$

for all $B \in S_k$ and for all $n \geq k$. 

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It remains to show that $\rho$ satisfies (H1), (H2) and (H3). We start by verifying (H1). Clearly $\rho(B) \geq \hat{\rho}(B) \geq \eta_-$ for all $B \in S_k$. On the other hand, (E) implies that $\omega_B \leq (1 - \delta)^{1/\beta}$, since

$$\sum_{B' \in D_{k+1}(B) \cap N} \omega^\beta_B C(B') \leq \sum_{B' \in D_{k+1}(B)} \rho(B')^\beta C(B') = C(B) \leq (1 - \delta) \sum_{B' \in D_{k+1}(B) \cap N} C(B').$$

This combined with $\sigma \leq \frac{1}{4}$ and property (3) of $\hat{\rho}$ implies that

$$\eta_- \leq \rho(B) \leq \frac{1}{2} \lor (1 - \delta)^{1/\beta}.$$

By setting $\eta_+ = (1/2) \lor (1 - \delta)^{1/\beta} \in (0, 1)$, we obtain (H1).

Since $\rho \geq \hat{\rho}$, $\rho$ satisfies (H3'). Therefore by Proposition 3.25 it suffices to show (H2). Let $B \sim B' \in S_k$, $k \geq 1$. We consider two cases.

Case 1: $g(B)_{k-1} = g(B')_{k-1}$. Then $\frac{\pi(B)}{\pi(B')} = \frac{\rho(B)}{n(B')}$ which thanks to (H1) satisfies

$$\eta_- \leq \frac{\pi(B)}{\pi(B')} \leq \eta_-^{-1}.$$

Case 2: $g(B)_{k-1} \neq g(B')_{k-1}$. Let $n \geq 0$ be the maximal integer such that $g(B)_n = g(B')_n$. In this case for $i = n + 1, \ldots, k$, we have $g(B)_i \sim g(B')_i$. Hence for $i = n + 2, \ldots, k$, $g(B)_i$ and $g(B')_i$ must both be peripheral (belong to $N^c$). Therefore

$$\frac{\pi(B)}{\pi(B')} = \frac{\rho(g(B)_{n+1})}{\rho(g(B')_{n+1})} \prod_{i=n+2}^{k} \frac{\rho(g(B)_i)}{\rho(g(B')_i)} = \frac{\rho(g(B)_{n+1})}{\rho(g(B')_{n+1})} \frac{\hat{\pi}(B)}{\hat{\pi}(B')} \frac{\hat{\rho}(g(B')_{n+1})}{\hat{\rho}(g(B)_{n+1})}.$$

By combining property (2) of $\hat{\rho}$ to estimate $\frac{\hat{\pi}(B)}{\pi(B')}$ and $\eta_- \leq \hat{\rho} \leq \rho \leq 1$ for the remaining terms, we obtain

$$\eta_-^3 \leq \frac{\pi(B)}{\pi(B')} \leq \eta_-^{-3}.$$

Combining the two cases, we obtain (H2) with constant $K_0 = \eta_-^{-3}$.

The following ‘patching lemma’ allows us to combine functions that satisfy local versions of (S1) and (S2) into a global one. This is an adaptation of the construction in [Car13, pp. 533–534].

**Lemma 3.28** (Patching lemma). Let $S$ denote a hyperbolic filling of a $K_D$-doubling, uniformly perfect, compact metric space, and let $\beta, \eta_1 > 0$. Let $S_d = (S, D_S)$ be a hyperbolic filling with horizontal parameter $\lambda \geq 8$ and vertical parameter $a > 1$ that satisfies (3.8). Let $C : S \to (0, \infty)$ be a $(K_h, K_v)$-gentle function. Assume that for all $B \in S_k$, $k \geq 1$, there exists $\sigma_B : S_{k+1} \to [0, \frac{1}{4})$ such that

(a) if we set $V_B = \{B' \in S_{k+1} : B' \cap 3 \cdot B \neq \emptyset\}$, then $\sigma_B(B') = 0$ for all $B' \in S_{k+1} \setminus V_B$.}

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(b) for any path $\gamma = \{B_i\}_{i=1}^N \in \Gamma_{k+1}(B)$, we have
\[
\sum_{i=1}^N \sigma(B_i) \geq 1,
\]
(c) and $\sum_{B' \in S_{k+1}} \sigma_B(B')^\beta \mathcal{C}(B') \leq \eta_1 \mathcal{C}(B)$.

Let $\sigma : S \to [0, \frac{1}{4})$ be defined as
\[
\sigma(B') = \max \{\sigma_A(B') : A \in S_k\}
\]
for all $B' \in S_{k+1}$ and for all $k \geq 1$, and $\sigma(B') = 0$ for all $B' \in S_0 \cup S_1$. Then there exists $C_{3.28} \geq 1$ that depends only on $K_D, K_h$ such that $\sigma$ satisfies (S1) and the estimate
\[
\sum_{B' \in D_{k+1}(B)} \sigma(B')^\beta \mathcal{C}(B') \leq C_{3.28} \eta_1 \mathcal{C}(B).
\]

Proof. For any path $\gamma = \{B_i\}_{i=1}^N \in \Gamma_{k+1}(B), B \in S_k$, we have $\sum_{i=1}^N \sigma(B_i) \geq \sum_{i=1}^N \sigma_B(B_i) \geq 1$. Therefore $\sigma$ satisfies (S1).

For any $B \in S_{k+1}$, we have
\[
\sum_{B' \in D_{k+1}(B)} \sigma(B')^\beta \mathcal{C}(B') = \sum_{B' \in D_{k+1}(B)} \max \{\sigma_A(B')^\beta : A \in S_k\} \mathcal{C}(B')
\]
\[
\leq \sum_{B' \in D_{k+1}(B)} \sum_{A : B' \in V_A} \sigma_A(B')^\beta \mathcal{C}(B')
\]
\[
\leq \sum_{A : V_B \cap V_A \neq \emptyset} \sum_{B' \in V_A} \sigma_A(B')^\beta \mathcal{C}(B')
\]
\[
\leq \sum_{A : V_B \cap V_A \neq \emptyset} \eta_1 \mathcal{C}(A)
\]
\[
\leq \sum_{A : A \sim B} \eta_1 \mathcal{C}(A) \quad (\because V_B \cap V_A \neq \emptyset \implies A \sim B)
\]
\[
\leq \sum_{A : A \sim B} \eta_1 K_h \mathcal{C}(B) \leq D_h K_h \eta_1 \mathcal{C}(B),
\]
where $D_h$ is chosen as (3.44). \hfill \Box

4 Universality of the conformal walk dimension

4.1 Consequences of Harnack inequalities

In this section, we recall some previous results concerning the elliptic and parabolic Harnack inequalities.

We recall the definition of heat kernel and the following sub-Gaussian estimate of the heat kernel.
Definition 4.1 (HKE(\(\beta\))). Let \((X, d, m, \mathcal{E}, \mathcal{F})\) be an MMD space, and let \(\{P_t\}_{t>0}\) denote its associated Markov semigroup. A family \(\{p_t\}_{t>0}\) of non-negative Borel measurable functions on \(X \times X\) is called the heat kernel of \((X, d, m, \mathcal{E}, \mathcal{F})\), if \(p_t\) is the integral kernel of the operator \(P_t\) for any \(t > 0\), that is, for any \(t > 0\) and for any \(f \in L^2(X, m)\),

\[
P_t f(x) = \int_X p_t(x, y) f(y) \, dm(y) \quad \text{for } m\text{-a.e. } x \in X.
\]

We say that \((X, d, m, \mathcal{E}, \mathcal{F})\) satisfies the heat kernel estimates HKE(\(\beta\)), if there exist \(C_1, c_1, c_2, \delta \in (0, \infty)\) and a heat kernel \(\{p_t\}_{t>0}\) such that for any \(t > 0\),

\[
p_t(x, y) \leq \frac{C_1}{m(B(x, t^{1/\beta}))} \exp(-c_1(d(x, y)^\beta/t^{1/(\beta-1)})) \quad \text{for } m\text{-a.e. } x, y \in X, \tag{4.1}
\]

\[
p_t(x, y) \geq \frac{c_2}{m(B(x, t^{1/\beta}))} \quad \text{for } m\text{-a.e. } x, y \in X \text{ with } d(x, y) \leq \delta t^{1/\beta}. \tag{4.2}
\]

Definition 4.2 (Capacity estimate). We say that \((X, d, m, \mathcal{E}, \mathcal{F})\) satisfies the capacity estimate \(\text{cap} (\beta)\) if there exist \(C_1, A_1, A_2 > 1\) such that for all \(R \in (0, \text{diam}(X, d)/A_2)\), \(x \in X\)

\[
C_1^{-1} \frac{m(B(x, R))}{R^\beta} \leq \text{Cap}(B(x, R), B(x, A_1 R) \setminus B(x, A_2 R)) \leq C_1 \frac{m(B(x, R))}{R^\beta}. \tag{4.3}
\]

The following lemma shows that the extended Dirichlet space is contained in the local Dirichlet space under the above heat kernel estimate.

Lemma 4.3. Let \((X, d, m, \mathcal{E}, \mathcal{F})\) be an MMD space that satisfies VD, RVD, and HKE(\(\beta\)) for some \(\beta \in [2, \infty)\). Then \(\mathcal{F}_e \subset \mathcal{F}_{loc}\).

Proof. By [GHL15, Theorem 1.2], we have the following Poincaré inequality: there exists \(C_P, A > 1\) such that

\[
\int_{B(x, r)} |f - f_{B(x, r)}|^2 \, dm \leq C_P r^\beta \int_{B(x, Ar)} \Gamma(f, f)
\]

for all \(f \in \mathcal{F}\), \(f_{B(x, r)} = \frac{1}{m(B(x, r))} \int_{B(x, r)} f \, dm\).

Let \(g \in \mathcal{F}_e\). Then there exists a \(\mathcal{E}\)-Cauchy sequence \(g_n\) such that \(g_n\) converges to \(g\) \(m\)-almost everywhere. For any ball \(B = B(x, r)\), by the Poincaré inequality the sequence \(g_n - (g_n)_{B}\) is \(L^2(B, m)\)-Cauchy. Since \(g_n\) converges to \(g\) almost everywhere and \(g_n - (g_n)_{B}\) is \(L^2(B, m)\)-Cauchy sequence, we have that \(\lim_{n \to \infty} (g_n)_{B} = g_{B}\) and that \(g_n\) converges to \(g\) in \(L^2(B, m)\) for any ball \(B\).

By [GHL15, Theorem 1.2], the following cutoff Sobolev inequality holds. There exists \(A_1, C_1, C_2\) for any \(x \in X, r < \text{diam}(X, d)/A_1\) there exists a function \(\phi \in \mathcal{F}\) such that \(\phi \equiv 1\) on a neighborhood of \(B(x, r)\), \(\phi\) is compactly supported on \(B(x, 2r)\), \(0 \leq \phi \leq 1\) such that

\[
\int_{B(x, 2r)} f^2 \, d\Gamma(\phi, \phi) \leq C_1 \int_{B(x, 2r)} d\Gamma(f, f) + \frac{C_2}{r^\beta} \int_{B(x, 2r)} f^2 \, dm \tag{4.3}
\]
for all \( f \in \mathcal{F} \). By a standard truncation argument, we may assume that \( g_n \) is bounded. By [FOT, Theorem 1.4.2(ii)], \( g_n \phi \in \mathcal{F} \). By Leibniz rule, the Cauchy Schwarz inequality, and strong locality, we obtain

\[
\mathcal{E}(\phi(g_n - g_m), \phi(g_n - g_m)) \leq 2 \left( \mathcal{E}(g_n - g_m, g_n - g_m) + \int_{B(x,2r)} (g_n - g_m)^2 d\Gamma(\phi, \phi) \right).
\]

Using (4.3) with \( f = g_n - g_m \), we conclude that \( g_n \phi \) is a Cauchy sequence with respect to the norm \( \cdot \mapsto (\mathcal{E}(f,f) + \|f\|^2_2)^{1/2} \). Clearly, the limit is \( g \phi \). Since \( g \phi = g \) in \( B(x,r) \), we conclude that \( g \in \mathcal{F}_{\text{loc}} \). □

We record the following theorem which relates the elliptic and parabolic Harnack inequality \( \text{EHI} \) and \( \text{PHI}(\beta) \). This theorem is due to Grigor’yan and Telcs in the context of random walks on graphs [GT02, Theorem 3.1]. This was later extended to MMD space setting by several authors.

**Theorem 4.4.** Let \((X,d,m,\mathcal{E},\mathcal{F})\) be an MMD space and let \( \beta \in [2,\infty) \). Then the following are equivalent.

(a) \((X,d,m,\mathcal{E},\mathcal{F})\) satisfies \( \text{PHI}(\beta) \).

(b) \((X,d,m,\mathcal{E},\mathcal{F})\) satisfies \( \text{VD}, \text{RVD}, \text{EHI} \) and \( \text{cap}(\beta) \).

(c) \((X,d,m,\mathcal{E},\mathcal{F})\) satisfies \( \text{VD}, \text{RVD}, \) and \( \text{HKE}(\beta) \).

**Proof.** The equivalence between (b) and (c) is contained in [GHL15, Theorem 1.2]. In [GHL15, Theorem 1.2], the condition \( \text{EHI} \) is stated for \( f \in \mathcal{F} \) but by using Lemma 4.3, we obtain the version in our definition, where \( f \in \mathcal{F}_e \).

The implication (b) implies \( \text{HKE}(\beta) \) follows from [GHL15, Theorem 1.2]. Assuming \( \text{VD} \), the equivalence between \( \text{PHI}(\beta) \) and \( \text{HKE}(\beta) \) is established in [BGK, Theorem 3.1]. This completes the proof of (b) implies (a).

For (a) implies (b), we first show this under the additional assumption \( \text{VD} \). By [GH, Proposition 5.6], \( \text{VD} \) and \( \text{EHI} \) imply that the space \((X,d)\) is uniformly perfect and hence \( m \) satisfies \( \text{RVD} \) by [Hei, Exercise 13.1]. The combination of \( \text{PHI}(\beta) \) and \( \text{VD} \) implies \( \text{HKE}(\beta) \) by [BGK, Theorem 3.1], and the combination of \( \text{VD}, \text{RVD}, \text{HKE}(\beta) \) and Lemma 4.3 implies \( \text{EHI} \) by [GT12, Theorem 7.4]. By [GHL15, Theorem 1.2], \( \text{VD}, \text{RVD} \) and \( \text{HKE}(\beta) \) implies \( \text{cap}(\beta) \). Therefore, it suffices to show that \( \text{PHI}(\beta) \) implies \( \text{VD} \).

The implication \( \text{PHI}(\beta) \) implies \( \text{VD} \) follows from [BGK, Theorem 3.2] under the additional assumption that the metric is geodesic. However, this additional assumption is not necessary and we modify the proof in [BGK] as follows. By [BGK, Lemma 4.6], there exists a heat kernel \( p_t(x,y) \) such that \((t,x,y) \mapsto p_t(x,y)\) is continuous in \((0,\infty) \times X \times X \). By [BGK, (4.52)], there exists \( c_1, c_2 > 0 \) such that

\[
\sup_{x,y \in B(x_0,r)} p_t(x,y) \geq \frac{c_1}{m(B(x_0,r))} \exp\left(-\frac{c_2 t}{r^\beta}\right) \quad \text{for all } x_0 \in X, r > 0, t > 0. \tag{4.4}
\]
Let $0 < C_1 < C_2 < C_3 < C_4$, $\delta \in (0, 1)$ and $C_5 > 1$ denote the constants in $\text{PHI}(\beta)$.

Define $K = \frac{C_4 + C_5}{C_4 + C_2} \in (1, \infty)$.

Let $x_0 \in X, r > 0$ be arbitrary. Fix $t > 0$ such that $t = (C_1 + C_2)\delta^{-\beta}r^{\beta}/2$. Using (4.4), we choose $y \in B(x_0, r)$ such that $\sup_{x \in B(x_0, r)} p_t(x, y) \geq \frac{1}{2} \frac{c_5}{2m(B(x_0, r))} \exp\left(-\frac{c_2 t}{r^\beta}\right)$. By $\text{PHI}(\beta)$, we obtain

$$p_{K^t}(x_0, y) \geq \frac{C_5^{-1}c_1}{2m(B(x_0, r))} \exp\left(-\frac{c_2 t}{r^\beta}\right) \quad \text{for some } y \in B(x_0, r).$$

By $\text{PHI}(\beta)$ for the caloric function $(t, z) \mapsto p_t(x_0, z)$ on the cylinder $(0, C_5\delta^{-\beta}r^{\beta}_1) \times B(x, \delta^{-1}r_1)$, where $r_1 > 0$ satisfies $(C_1 + C_2)\delta^{-\beta}r^{\beta}_1/2 = Kt$ (or equivalently, $r_1 = K^{1/\beta}r$) and (4.5), we obtain

$$p_{K^{2t}}(x_0, z) \geq \frac{C_5^{-2}c_1}{2m(B(x_0, r))} \exp\left(-\frac{c_2 t}{r^\beta}\right) \quad \text{for all } z \in B(x_0, K^{1/\beta}r).$$

Using $\int_X p_{K^{2t}}(x_0, z) m(dz) \leq 1$ and $t = (C_1 + C_2)\delta^{-\beta}r^{\beta}/2$ and (4.6), there exists $C_6 > 1$ such that

$$\frac{m(B(x_0, K^{1/\beta}r))}{m(B(x_0, r))} \leq C_6, \quad \text{for all } x_0 \in X, r > 0.$$

By iterating the above estimate $\lceil \beta \log 2/\log K \rceil$ times, we obtain the volume doubling property $\text{VD}$.

\[\square\]

**Remark 4.5.** Theorem 4.4 can be generalized to the case where the space-time scaling function $\Psi(r) = r^\beta$ is replaced with an arbitrary increasing, continuous function on $(0, \infty)$ satisfying the following estimate: there exist $C, \beta_1, \beta_2 > 0$ such that

$$C_1^{-1} \left(\frac{R}{r}\right)^{\beta_1} \leq \frac{\Psi(R)}{\Psi(r)} \leq C_1 \left(\frac{R}{r}\right)^{\beta_2}.$$ 

The generalized version of the relevant properties like $\text{PHI}(\beta)$ and $\text{cap}(\beta)$ for such space-time scale functions can be found in [BGK, GHL15].

As mentioned in the introduction, Delmotte constructs a space that satisfies EHI but fails to satisfy $\text{VD}$ and hence fails to satisfy $\text{PHI}(\beta)$ for any $\beta$ [Del]. Nevertheless, it is possible to obtain $\text{PHI}(\beta)$ after a time change and change of metric. We recall the characterization of elliptic Harnack inequality in [BM, BCM].

**Theorem 4.6.** ([BM, BCM]) Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be a complete, doubling, locally compact MMD space. Then the following are equivalent:

1. $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfies the EHI.

2. There exists $\gamma > 2$, $\mu \in \mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F})$ and $\theta \in \mathcal{J}(X, d)$ such that the time changed MMD space $(X, \theta, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu)$ satisfies $\text{PHI}(\gamma)$. In other words, $d_{cw} < \infty$. 

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Either of the two equivalent conditions imply that \((X, d)\) is uniformly perfect.

**Proof.** This follows from [BCM, Theorem 7.9], Theorem 4.4, and [GHL15, Theorem 1.2]. □

The following elementary lemma is used to verify that the function defined in (4.7) on a hyperbolic filling is gentle and satisfies the enhanced subadditive estimate (E).

**Lemma 4.7.** Let \((X, d, m)\) be a metric measure space that satisfies \(\text{VD}\) and let \(\gamma > 0\). For any ball \(B(x, r)\), we define

\[
C(B(x, r)) = \frac{m(B(x, r))}{r^\gamma}.
\]

(4.7)

(a) Let \(\lambda \geq 1\). There exists \(C_1 > 0\) (that depends on the only the constant of \(\text{VD}\) and \(\lambda\)) such that for any \(x, y \in X\) satisfying \(B(x, \lambda r) \cap B(y, \lambda r) \neq \emptyset\), we have

\[
C(B(x, r)) \leq C_1 C(B(y, r)).
\]

(b) Let \(a > 1\). There exists \(C_2 \geq 1\) (that depends on the only the constant of \(\text{VD}\), \(\gamma\) and \(\lambda\)) such that for any \(x, y \in X\) satisfying \(y \in B(x, r)\), we have

\[
C_2^{-1} C(B(y, r/a)) \leq C(B(x, r)) \leq C_2 C(B(y, r/a)).
\]

(c) There exists \(C_3 > 1\) such that the following estimate holds: for all \(a > 1, x \in X, r > 0\) and \(z_1, \ldots, z_k, k \in \mathbb{N}\) such that \(d(z_i, z_j) \geq r/(2a)\) for all \(1 \leq i < j \leq k\) and satisfying \(\bigcup_{i=1}^k B(z_i, r/a) \supset B(x, r/6)\), we have that

\[
C(B(x, r)) \leq C_3 a^{-\gamma} \sum_{i=1}^k C(B(z_i, r/a)).
\]

(4.8)

**Proof.** We denote \(m(B(x, r))\) by \(V(x, r)\) in this proof.

(a) Let \(C_D \in (1, \infty)\) denote the constant associated with \(\text{VD}\) and let \(\alpha = \log_2 C_D\). By using the volume doubling finitely for balls of the form \(B(x, 2^i r), i \in \mathbb{Z}_{\geq 0}\), we obtain

\[
\frac{V(x, R)}{V(x, r)} \leq C_D \left(\frac{R}{r}\right)^\alpha, \text{ for all } 0 < r \leq R \text{ and } x \in X.
\]

(4.9)

Let \(z \in B(x, \lambda r) \cap B(y, \lambda r)\). By using \(B(x, r) \subset B(z, (\lambda + 1)r)\), \(B(z, r) \subset B(y, (\lambda + 1)r)\) and (4.9), we obtain

\[
V(x, r) \leq V(z, (\lambda + 1)r) \leq C_D(\lambda + 1)^\alpha V(z, r) \leq C_D^2(\lambda + 1)^{2\alpha} V(y, r).
\]
(b) Since $B(x, r) \subset B(y, 2r)$ and $B(y, r/a) \subset B(x, 2r)$, by (4.9) we have

$$V(x, r) \leq V(y, 2r) \leq C_D(2a)^\alpha V(y, r/a), \quad V(y, r/a) \leq V(x, 2r) \leq C_D V(x, r).$$

Therefore

$$C(B(x)) \leq C_D 2^\alpha a^{\alpha - \gamma} C(B(y, r/a)), \quad C(B(y, r/a)) \leq C_D a^\gamma C(B(x)).$$

(c) By VD and $\cup_{i=1}^k B(z_i, r/a) \supset B(x, r/3)$, we have

$$V(x, r) \leq C_3^3 V(x, r/8) \leq C_3^2 \sum_{i=1}^k V(x_i, r/a).$$

Dividing both sides by $r^\gamma$, we obtain (4.8) with $C_3 = C_D^3$.

□

The elliptic Harnack inequality implies that the capacities across annuli with similar locations and scales are comparable as we recall below.

**Lemma 4.8.** ([BCM, Lemma 5.21 and 5.22]) Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space that satisfies EHI, where $(X, d)$ is a doubling metric space. For any $A_1, A_2 > 1$, there exists $C_1, C_2 > 1$ and $\gamma > 0$ such that for all $x, \tilde{x} \in X$, and for any $0 < s \leq r < \text{diam}(X, d)/C_1$, we have

$$C_2^{-1} \left( \frac{r}{s} \right)^{-\gamma} \leq \frac{\text{Cap}(B(x, r), B(x, A_2 r)^c)}{\text{Cap}(B(x, s), B(x, A_1 s)^c)} \leq C_2 \left( \frac{r}{s} \right)^{\gamma}.$$  

**Proof.** This follows immediately from [BCM, Theorem 5.4, Lemmas 5.21 and 5.22].  □

Using this lemma, we obtain the following comparison of capacity across annuli under a quasisymmetric change of metric.

**Proposition 4.9.** Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space that satisfies PHI($\gamma$), where $\gamma \geq 2$. Let $\theta \in \mathcal{F}(X, d)$ and $a > 1$. Then there exists $C, A > 0$ such that the following property holds. For any $x, \tilde{x} \in X, 0 < r < \text{diam}(X, \theta)/A$, $s > 0$, $n \in \mathbb{Z}$ such that

$$s = \sup \{ 0 < t < 2\text{diam}(X, d) : B_d(x, t) \subset B_\theta(x, r) \}, \quad (4.10)$$

and

$$2a^{-n-1} \leq s < 2a^{-n}, \quad d(\tilde{x}, x) < 2a^{-n},$$

we have

$$C^{-1} \frac{m(B_d(\tilde{x}, 2a^{-n}))}{[2a^{-n}]^\gamma} \leq \text{Cap}(B_\theta(x, r), B_\theta(x, 2r)^c) \leq C \frac{m(B_d(\tilde{x}, 2a^{-n}))}{[2a^{-n}]^\gamma}. \quad (4.11)$$
Proof. By Theorem 4.4, $(X,d)$ satisfies doubling, uniformly perfect metric space. By Proposition 3.2, there exists $A_1,A_2,A_3 > 1$ such that for all $x \in X, 0 < r < \text{diam}(X,\theta)$,

$$B_d(x, s) \subset B_\sigma(x, r) \subset B_d(x, A_1 s) \subset B_d(s, 2A_1 s) \subset B_\sigma(x, A_2 r) \subset B_d(x, A_3 s),$$  

where $s > 0$ is as defined in (4.10). If $B_d(x, A_3 s) \neq X$ in (4.12), we have

$$\text{Cap}(B_d(x, s), B(x, A_3 s)^c) \leq \text{Cap}(B_\sigma(x, r), B_\sigma(x, A_2 r)^c) \leq \text{Cap}(B_d(x, A_1 s), B(x, 2A_1 s)^c).$$

By Lemma 4.8, Proposition 3.2(b), and Theorem 4.4, there exist $C_1,A > 1$ such that for all $x \in X, 0 < r < \text{diam}(X,\theta)/A$, we have

$$C_1^{-1} \text{Cap}(B_d(x, s), B(x, A_3 s)^c) \leq \text{Cap}(B_\sigma(x, r), B_\sigma(x, 2r)^c) \leq C_1 \text{Cap}(B_d(x, s), B(x, A_3 s)^c),$$

and

$$C_1^{-1} \frac{m(B_d(x, s))}{s^\gamma} \leq \text{Cap}(B_d(x, s), B(x, A_3 s)^c) \leq C_1 \frac{m(B_d(x, s))}{s^\gamma},$$

where $s > 0$ is as given in (4.10). By (4.14), (4.15) and VD, we obtain (4.11). \hfill \Box

We will use Theorem 3.24 to construct metrics. The following proposition plays a central role in constructing a function on the hyperbolic filling that satisfies the hypotheses (S1) and (S2) in Theorem 3.24.

Proposition 4.10. Let $(X,d,m,\mathcal{E},\mathcal{F})$ be an MMD space that satisfies $\text{PHI}(\gamma)$ for some $\gamma > 2$ and let $\lambda > 1$. There exists constants $A,C_1,C_2 > 1,\eta > 0$ (that depend only on $\lambda$ and the constants associated with $\text{PHI}(\gamma)$) such that for any $a > 1, x \in X, 0 < r < \text{diam}(X,d)/A$, and for any collection of balls $B = \{B(y_i, r/a) : i \in I\}$ such that $\bigcup_{i \in I} B(y_i, r/a) = X$ and $\{B(y_i, r/(4a))\}$ is pairwise disjoint, there exists a function $\sigma : \mathcal{B} \to [0,\infty)$ that obeys the following properties (note that $\sigma$ depends on $x \in X, r > 0$):

(S1') for any sequence of balls $\gamma = \{B_i : 1 \leq i \leq N\}$ in $\mathcal{B}$ such that $x_{B_i} \in B(x, r), x_{B_N} \notin B(x, 2r)$ and $\lambda \cdot B_i \cap \lambda \cdot B_{i+1} \neq \emptyset$ for all $i = 1,\ldots,N-1$, we have

$$\sum_{i=1}^{N} \sigma(B_i) \geq 1,$$  

and

$$\sigma(B) = 0, \quad \text{for any ball } B \in \mathcal{B} \text{ such that } x_B \notin B(x, 2r).$$

(S2') $\sigma : \mathcal{B} \to (0,\infty)$ satisfies the following estimates

$$\sum_{B \in \mathcal{B}} \sigma(B)^2 \frac{m(B)}{(r/a)^\gamma} \leq C_1 \frac{m(B(x, r))}{r^\gamma},$$

and

$$\sup_{B \in \mathcal{B}} \sigma(B) \leq C_2 a^{-\eta}.$$
In particular, for any $\beta > 2$, we have
\[
\sum_{B \in \mathcal{B}} \sigma(B) \beta \frac{m(B)}{(r/a)^{\gamma}} \leq C_1 C_2 \beta^{2} a^{-(\beta - 2)} \eta \frac{m(B(x, r))}{r^{\gamma}}. \quad (4.20)
\]

**Proof.** For a function $u \in C(X) \cap \mathcal{F}$ and $B \in \mathcal{B}$, we define its ‘discretization’ $u_d : \mathcal{B} \to \mathbb{R}$ as
\[
u_d(B) := \int_B u \, dm = \frac{1}{m(B)} \int_B u \, dm, \quad (4.21)
\]
and its ‘discrete gradient’ $\sigma_u : \mathcal{B} \to [0, \infty)$
\[
\sigma_u(B) := \sum_{B' \in \mathcal{B} : B' \cap \lambda \cdot B \neq \emptyset} |u_B(B') - u_B(B)|. \quad (4.22)
\]

Our construction of $\sigma$ is the discrete gradient $\sigma_u$ of a well chosen function $u$. In particular, we choose a function $u \in C_c(X) \cap \mathcal{F}$ that satisfies the following properties: there exists $C_3 > 1, \eta > 0$ (that depends only on the constant associated with PHI($\gamma$) such that for all $x \in X, r < \text{diam}(X, d)/A$, we have
\[
u \equiv 1 \text{ on } B(x, 1.1r) \text{ and } \nu \equiv 0 \text{ on } B(x, 1.9r) \quad (4.23)
\]
\[
\mathcal{E}(u, u) \leq C_3 \frac{\mu(B(x, r))}{r^{\gamma}}, \quad (4.24)
\]
\[
|u(y) - u(z)| \leq C_3 \left( \frac{d(y, z)}{r} \right)^{\eta} \text{ for all } y, z \in X. \quad (4.25)
\]

The existence of a function $u \in C_c(X) \cap \mathcal{F}$ satisfying the above properties follows from the cutoff Sobolev inequality in [BBK, Definition 2.5] and a standard covering argument as we recall below\(^1\). By Theorem 4.4 we have that $m$ is a doubling measure in $(X, d)$ and hence $(X, d)$ is $K_D$-doubling metric space for some $K_D > 1$. Therefore there exists $N_D \in \mathbb{N}$ that depends only on $K_D$ and $y_1, \ldots, y_{N_D} \in B(x, 1.1r)$ such that $\bigcup_{i=1}^{N_D} B(y_i, r/10) \supset B(x, 1.1r)$. By the construction cutoff functions in [BBK, Section 3], there exists $C_4 > 0, \eta > 0$ such that for each $i = 1, \ldots, N_D$ satisfies
\[
\phi_i \equiv 1 \text{ on } B(y_i, r/10), \quad \phi_i \equiv 0 \text{ on } B(y_i, r/5)^c,
\]
\[
\mathcal{E}(\phi_i, \phi_i) \leq C_4 \frac{m(B(y_i, r/10))}{r^{\gamma}}, \quad |\phi_i(y) - \phi_i(z)| \leq C_4 \left( \frac{d(y, z)}{r} \right)^{\eta} \text{ for all } y, z \in X.
\]

By choosing $u = \max_{1 \leq i \leq N_D} \phi_i$ and using the above estimates along with triangle inequality, $\mathcal{E}(u, u) \leq \sum_{i=1}^{N_D} \mathcal{E}(\phi_i, \phi_i), |u(y) - u(z)| \leq \max_{1 \leq i \leq N_D} |\phi_i(y) - \phi_i(z)|$, we obtain the desired properties $(4.23),(4.24)$ and $(4.25)$.

\(^1\)We note that the proof of cutoff Sobolev inequality in [BBK, Lemma 3.3] has a gap which has been resolved in the arXiv version. The proof of cutoff Sobolev inequality also works in the compact setting with minor modifications (one needs $r < \text{diam}(X, d)/A$ for a large enough $A$ for the compact case) and it does not use the assumption that metric is geodesic.
Let us show that the function $\sigma = \sigma_u$ as defined by (4.21) and (4.22) satisfies the desired conditions (S1') and (S2'). To this end, we note the following properties of $u_d : B \to \mathbb{R}$:

$$u_d(B(x_B, r/10)) = 1 \quad \text{for any } x_B \in B(x, r/10) \quad \text{(since } B(x_B, r/10) \subset B(x, 1.1r)),$$

$$u_d(B(y, r/a)) = 0 \quad \text{for any } y \in X \text{ such that } d(y, y') \leq 2\lambda r/a \text{ where } y' \in B(x, 2r)^c \quad (B(y, r/a) \subset B(x, 1.9r)^c \text{ because } (2\lambda + 1)r/a < 0.1r \text{ by (3.11)}),$$

$$\sum_{i=1}^{N} \sigma_u(B_i) \geq \sum_{i=1}^{N-1} |u_d(B_i) - u_d(B_{i+1})| \geq |u_d(B_1) - u_d(B_N)|$$

for any sequence of balls $B_1, \ldots, B_N \in \mathcal{B}$ such that $\lambda \cdot B_i \cap \lambda \cdot B_{i+1} \neq \emptyset$ for all $i = 1, \ldots, N - 1$. The above equations immediately imply (4.16) and (4.17).

Since the balls $B(y_i, r/(4a)), i \in I$ are disjoint, by doubling property of $(X, d)$, there exists $C_5 > 1$ that depends only on $\lambda$ and the doubling constant (but not on $a$) such that

$$\# \{ B' \in \mathcal{B} : \lambda \cdot B \cap \lambda \cdot B' \neq \emptyset \} \leq C_5, \quad \text{for all } B \in \mathcal{B}. \quad (4.26)$$

For any two balls $B, B' \in \mathcal{B}$ such that $\lambda \cdot B \cap \lambda \cdot B' \neq \emptyset$, by (4.25) we have

$$|u_d(B) - u_d(B')| \leq \sup_{y, z \leq 2(\lambda + 1)r/a} |u(y) - u(z)| \leq C_3(2(\lambda + 1)^n a^{-n}). \quad (4.27)$$

Combining (4.22), (4.26) and (4.27), we obtain (4.19) for $\sigma = \sigma_u$.

It remains to show that $\sigma = \sigma_u$ satisfies (4.19). To this end, we recall the following Poincaré inequality (follows from [GHL15, Theorem 1.2] and Theorem 4.4): there exist $C_P, A > 1$ such that

$$\frac{1}{2m(B(y, s))} \int_{B(y, s)} \int_{B(y, s)} (f(z) - f(w))^2 \, dm(z) \, dm(w) \leq C_P \int_{B(y, As)} d\Gamma(f, f), \quad (4.28)$$

for any $f \in \mathcal{F}, y \in X, s > 0$. Using this Poincaré inequality, the following comparison estimate between discrete and continuous energies is standard [CS, BB04]. Similar to §3.2, for any two balls $B, B' \in \mathcal{B}$ by $B' \sim B$ we mean that $\lambda \cdot B \cap \lambda \cdot B' \neq \emptyset$. We obtain
(4.18) by the following estimates:

\[
\sum_{B \in B} \sigma_u^2(B) \frac{m(B)}{(r/a)\gamma} \lesssim \sum_{B,B' \in B, B' \sim B} |u_d(B') - u_d(B)|^2 \frac{m(B)}{(r/a)\gamma} \quad \text{(by (4.26) and the Cauchy-Schwarz inequality)}
\]

\[
\lesssim \sum_{B \in B} \frac{1}{(r/a)\gamma m(B')} \int_B \int_{B'} (u(y) - u(z))^2 \, dm(y) \, dm(z) \quad \text{(by Jensen’s inequality)}
\]

\[
\lesssim \sum_{B \in B} \frac{1}{m((2\lambda + 1) \cdot B)(r/a)\gamma} \int_B \int_{(2\lambda+1) \cdot B} (u(y) - u(z))^2 \, dm(y) \, dm(z) \quad \text{(by VD)}
\]

\[
\lesssim \sum_{B \in B} (2\lambda + 1)^\gamma \int_{A(2\lambda+1) \cdot B} d\Gamma(u, u) \quad \text{(by 4.28)}
\]

\[
\lesssim \mathcal{E}(u, u) \quad \text{(since (X, d) is K}_D\text{-doubling, we have } \sum_{B \in S_n} 1_{(2\lambda\lambda + A) \cdot B} \lesssim 1).
\]

\[
\lesssim \frac{m(B(x, r))}{r^\gamma} \quad \text{(by (4.24)).}
\]

Finally, (4.20) follows from (4.18), (4.19), and

\[
\sum_{B \in B} \sigma(B)^\beta \frac{m(B)}{(r/a)\gamma} \leq \left( \sup_{B \in B} \sigma(B) \right)^{\beta-2} \sum_{B \in B} \sigma(B)^2 \frac{m(B)}{(r/a)\gamma}.
\]

The following proposition provides a convenient sufficient conditions for a measure \( \mu \) to be smooth and have full quasi-support.

**Proposition 4.11.** Let \((X, d, m, \mathcal{E}, \mathcal{F})\) be an MMD space that satisfies \( \text{PHI}(\gamma) \) for some \( \gamma \geq 2 \) and let \( \theta \in J(X, d) \). Let \( \beta > 2 \) and \( \mu \) be a measure that satisfies the following estimate: there exists \( C_1, A > 1 \) such that for any \( x \in X, 0 < r < \text{diam}(X, \theta)/A \), we have

\[
C_1^{-1} \frac{\mu(B_\theta(x, r))}{r^\beta} \leq \text{Cap}(B_\theta(x, r), B_\theta(x, 2r)^\gamma) \leq C_1 \frac{\mu(B_\theta(x, r))}{r^\beta}.
\]

Then \( \mu \) is a smooth measure with full quasi-support; or equivalently \( \mu \in \mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F}) \). Furthermore \( \mu \) satisfies \( \text{VD} \) and \( \text{RVD} \) on \((X, \theta)\).

**Proof.** By [BM, Lemma 6.3], the MMD space \((X, d, m, \mathcal{E}, \mathcal{F})\) satisfies \( \text{EHI} \). Since \((X, d)\) is a doubling, uniformly perfect metric space, so is \((X, \theta)\) [Hei, Theorem 10.18 and Exercise 11.2]. The volume doubling property \( \text{VD} \) of \( \mu \) in \((X, \theta)\) follows from [BCM, Lemma 6.3]. The \( \text{RVD} \) property for \( \mu \) follows from the fact that \((X, \theta)\) is uniformly perfect [Hei, Exercise 13.1]. That \( \mu \) is a smooth measure follows from [BCM, Proposition 6.17 and Theorem 4.6]. By [BCM, Proposition 5.17 and Theorem 5.4], \( \mu \) has full quasi support. \( \square \)
4.2 Proof of Theorem 2.10.

Proof. By Theorem 4.6, it suffices to show (a) implies (b). To this end, we fix an arbitrary $\beta > 2$. We shall construct a metric $\theta \in \mathcal{J}(X,d)$ and a measure $\mu \in \mathcal{A}(X,d,m,\mathcal{E},\mathcal{F})$ such that the time changed MMD space $(X,\theta,\mu,\mathcal{E}^\mu,\mathcal{F}^\mu)$ satisfies $\text{PHI}(\beta)$.

By Theorem 4.6, (1.2), (2.7), and by change the metric and measure if necessary, we may assume that $(X,d,m,\mathcal{E},\mathcal{F})$ satisfies $\text{PHI}(\gamma)$ where $\gamma > 2$. By Theorem 4.4, we have $(X,d,m,\mathcal{E},\mathcal{F})$ satisfies VD, RVD, EHI and $\text{cap}(\beta)$.

If $(X,d)$ is bounded, we scale the metric so that $\text{diam}(X,d) = \frac{1}{2}$.

Fix $\lambda \geq 32$ and let $a$ be an arbitrary constant that satisfies (3.11). The choice of $a$ will be made later in the proof. Let $x_0 \in X$. Let $S = \bigsqcup_{k \in \mathbb{Z}_+} S_k$ denote the vertex set of the hyperbolic filling as defined in Definition 3.10, where $S_k = \{B(x,2a^{-k}) : x \in N_k\}$, where $N_k,k \in \mathbb{Z}$ is a sequence of $a^{-k}$-separated sets such that $N_k \supset N_{k+1}$ and $x_0 \in N_k$ for all $k \in \mathbb{Z}$.

We define a function $C : \bigsqcup_{k \in \mathbb{Z}} S_k \to (0,\infty)$ on the extended hyperbolic filling by

$$C(B(x,2a^{-k})) = \frac{m(B(x,2a^{-k}))}{(2a^{-k})^{\gamma}},$$

for any $k \in \mathbb{Z}$ and for any $B(x,2a^{-k}) \in S_k$. (4.29)

Let us verify that $C$ is gentle and satisfies the enhanced subadditivity property (E). By Lemma 4.7(a,b), there exist $K_h,K_v$ such that $K_v$ depends only on $a$ and the constant associated with VD, $K_h$ depends only on $\lambda$ and the constant associated with VD such that

$$C(B_1) \leq K_h C(B_2), \text{ whenever } B_1 \text{ and } B_2 \text{ share a horizontal edge},$$

$$C(B_1) \leq K_v C(B_2), \text{ whenever } B_1 \text{ and } B_2 \text{ share a vertical edge}. \quad (4.30)$$

Recall that for every ball $B \in S_k,k \in \mathbb{Z}$, there exists an unique ball $g(B)_{k-1} \in S_{k-1}$ such that there is a vertical edge between $g(B)_{k-1}$. We say that a ball $B \in S_k,k \in \mathbb{Z}$ is non-peripheral if

$$B \sim B' \text{ implies that } g(B)_{k-1} = g(B')_{k-1}.$$  

Note that is $B \sim B'$ and $B,B' \in S_{k+1}$, then by (3.11), we have $d(x_B,x_{B'}) \leq 2\lambda a^{-k-1} \leq \frac{1}{12} a^{-k}$. We denote the set of all non-peripheral elements of $\bigsqcup_{k \in \mathbb{Z}} S_k$ by $\mathcal{N}$. Hence if $C = g(B)$ and $d(x_C,x_B) < \frac{1}{6} (2a^{-k} + 2a^{-k-1})$, then $d(x_C,x_{B'}) < (3^{-1} + 2a^{-1} + 12^{-1})a^{-k} < \frac{1}{2} a^{-k}$, and hence $B \in \mathcal{N}$. This along with Lemma 4.7(c) imply that, there exists $C_1 > 0$ such that

$$C(B) \leq C_1 a^{-\gamma} \sum_{B' \in \mathcal{N} \cap \mathcal{D}_{k+1}(B)} C(B'), \text{ for all } B \in S_k,k \in \mathbb{Z}. \quad (4.31)$$

For $k \in \mathbb{Z}$ and $B \in S_k$, let $\Gamma_{k+1}(B)$ denote the set of horizontal paths $\gamma = \{B_i\}_{i=1}^N, N \geq 2$ such that $B_i \in S_{k+1}$ for all $i = 1, \ldots, N$, $B_i \sim B_{i+1}$ for all $i = 1, \ldots, N-1$, $x_{B_1} \in B$, $x_{B_N} \notin 2 \cdot B$, and $x_{B_i} \in 2 \cdot B$ for all $i = 1, \ldots, N-1$ (as given in Definition 3.22). If $\text{diam}(X,d) = \frac{1}{2}$, we note that $\Gamma_k(B) = 0$ for all $k \leq 0$. If $\text{diam}(X,d) = \frac{1}{2}$, we assume that

$$a > 2A, \quad (4.32)$$

For $k \in \mathbb{Z}$ and $B \in S_k$, let $\Gamma_{k+1}(B)$ denote the set of horizontal paths $\gamma = \{B_i\}_{i=1}^N, N \geq 2$ such that $B_i \in S_{k+1}$ for all $i = 1, \ldots, N$, $B_i \sim B_{i+1}$ for all $i = 1, \ldots, N-1$, $x_{B_1} \in B$, $x_{B_N} \notin 2 \cdot B$, and $x_{B_i} \in 2 \cdot B$ for all $i = 1, \ldots, N-1$ (as given in Definition 3.22). If $\text{diam}(X,d) = \frac{1}{2}$, we note that $\Gamma_k(B) = 0$ for all $k \leq 0$. If $\text{diam}(X,d) = \frac{1}{2}$, we assume that

$$a > 2A, \quad (4.32)$$
where $A$ is the constant in Proposition 4.10. If either $k \in \mathbb{Z}, \text{diam}(X, d) = \infty$ or if $k \in \mathbb{N}, \text{diam}(X, d) = \frac{1}{2}$, for any $B \in \mathcal{S}_k$, we define $\sigma_B : \mathcal{S}_{k+1} \to (0, \infty)$ as the function defined in Proposition 4.10, that satisfies

$$\sum_{i=1}^{N} \sigma_B(B_i) \geq 1, \quad \text{for any } \{B_i\}_{i=1}^{N} \in \Gamma_{k+1}(B).$$

Otherwise if $\text{diam}(X, d) = \frac{1}{2}$ and $k \geq 0$, we simply define $\sigma_B : \mathcal{S}_{k+1} \to [0, \infty)$ as $\sigma_B \equiv 0$ for all $B \in \mathcal{S}_k$. For any $k \in \mathbb{Z}$ and for any $B \in \mathcal{S}_k$, we define

$$\sigma(B) = \max_{C \in \mathcal{S}_{k-1}} \sigma_C(B), \quad \text{for any } k \in \mathbb{Z}, B \in \mathcal{S}_k. \quad (4.33)$$

Evidently, by Proposition 4.10, we have

$$\sum_{i=1}^{N} \sigma(B_i) \geq 1, \quad \text{for any } \{B_i\}_{i=1}^{N} \in \Gamma_{k+1}(B) \text{ and for any } k \in \mathbb{Z}, B \in \mathcal{S}_k. \quad (4.34)$$

In the compact case, the above statement is vacuously true for $k \leq 0$. By Proposition 4.10 and the argument in Lemma 3.28, there exist $C_2, \eta > 0$ such that

$$\sum_{B' \in D_{k+1}(B)} \sigma(B')^\beta \mathcal{C}(B') \leq C_2 a^{-(\beta-2)\eta} \mathcal{C}(B), \quad \text{for any } k \in \mathbb{Z}, B \in \mathcal{S}_k, \quad (4.35)$$

where $D_{k+1}(B)$ denote the set of descendants of $B$ in $\mathcal{S}_{k+1}$ (that is, $D_{k+1}(B)$ is the set of elements in $\mathcal{S}_{k+1}(B)$ that share a vertical edge with $B$).

We consider two cases.

**Case 1:** $(X, d)$ is bounded. Let $\mathcal{S} = \coprod_{k \geq 0} \mathcal{S}_k$ denote the vertex set of the hyperbolic filling. In this case by (4.31), we can ensure the enhanced subadditivity estimate (E) by choosing $a$ large enough. Similarly by (4.34) and (4.35), the function $\sigma$ defined above satisfies the hypotheses (S1) and (S2) of Theorem 3.24 for all large enough $a$. Therefore by Theorems 3.24 and 3.14, and Proposition 3.21, there exist a metric $\theta \in \mathcal{J}(X, d)$ and a measure $\mu$ on $X$ that satisfies the following estimate: there exists $C_3 > 0$ such that

$$C_3^{-1} r^\beta \mathcal{C}(B) \leq \mu(B_\theta(x, r)) \leq C_3 r^\beta \mathcal{C}(B), \quad \text{for all } x \in X, r < \text{diam}(X, \theta), B \in \mathcal{A}_S(B_\theta(x, s)), \quad (4.36)$$

where $s$ is the largest number in $[0, 2\text{diam}(X, d)]$ such that $B_\theta(x, s) \subset B_\theta(x, r)$ (as defined in (3.2)) and $\mathcal{A}_S(B_\theta(x, s))$ is as given in Definition 3.16. Combining (4.36) and Proposition 4.9, there exist $A_1, C_4 > 0$ such that

$$C_4^{-1} \frac{\mu(B_\theta(x, r))}{r^\beta} \leq \text{Cap}(B_\theta(x, r), B_\theta(x, 2r)^c) \leq C_4 \frac{\mu(B_\theta(x, r))}{r^\beta}, \quad (4.37)$$

for any $x \in X, 0 < r < \text{diam}(X, d)/A_1$. By Proposition 4.11, $\mu \in \mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F})$, $\mu$ has full quasi support, and $\mu$ satisfies VD and RVD on $(X, \theta)$. ****(need to justify EHI under time change more carefully)By Theorem 4.4, (4.37), [BM, Lemma 5.3], and Proposition
we have that the MMD space \((X, \theta, \mu, \mathcal{E}^n, \mathcal{F}^n)\) satisfies \(\Phi(\beta)\). Since \(\beta > 2\) is arbitrary, we conclude that the conformal walk dimension is two.

**Case 2:** \((X, d)\) is unbounded.

The approach in the unbounded case is to construct metrics and measures on an increasing sequence of compact sets that cover \(X\), and to take suitable sub-sequential limit. Let \(x_0 \in X\) be the point such that \(x_0 \in N_k\) for all \(k \in \mathbb{Z}\) as given in Definition 3.10. We consider the sequence of subsets

\[
X_n = \overline{D_{-n}(x_0)}, \quad \text{for any } n \in \mathbb{N},
\]

where \(D_{-n}(x_0)\) is as defined in (3.15). By Lemma 3.9(c), \(X_n\) is compact and satisfies

\[
B(x_0, (2^{-1} - (a - 1)^{-1})a^n) \subset X_n \subset \overline{B(x_0, (1 - a^{-1})^{-1}a^n)}.
\]

For any \(k \geq -n\), we define

\[
S_k^{(n)} = \{ B(x, 2a^{-k}) \cap X_n : x \in N_k \cap D_{-n}(x_0) \}.
\]

By Lemma 3.9, \(S^{(n)} := \bigsqcup_{k \in \mathbb{Z}, k \geq -n} S_k^{(n)}\) is a hyperbolic filling of the compact space with the same vertical edges induced from the extended hyperbolic filling. Similarly, \(B \cap X_n, B' \cap X_n \in S_k^{(n)}\) share a horizontal edge if and only if \((\lambda \cdot B \cap X_n) \cap (\lambda \cdot B' \cap X_n) \neq \emptyset\). We define \(C_n : S^{(n)} \to (0, \infty), \sigma_n : S^{(n)} \to [0, \infty)\) as

\[
C_n(B \cap X_n) = C(B), \quad \sigma_n(B \cap X_n) = \sigma(B),
\]

for any \(B \in B \in \bigsqcup_{k \geq -n} S_k\), where \(C\) is as given in (4.29) and \(\sigma\) is a \(s\) given in (4.33). Similar to the compact case, by choosing \(a > 1\) large enough, by (4.31), we obtain the enhanced subadditivity estimate (E) for \(C_n\) uniformly over \(n\) (that is, the constant \(\delta\) in associated with (E) does not depend on \(n\)) Similarly, by increasing \(a\) is necessary, and by (4.34) and (4.35), the function \(\sigma_n\) defined above satisfies the hypotheses (S1) and (S2) of Theorem 3.24 uniformly in \(n\) (that is, the constant \(\eta_0\) in associated with (S2) does not depend on \(n\)).

Similar to the compact case, by Theorems 3.24 and 3.14, there exist metrics \(\theta_n \in \mathcal{J}(X_n, d)\) for each \(n \in \mathbb{N}\), and a distortion function \(\eta : [0, \infty) \to [0, \infty)\) such that the identity map

\[
\text{Id} : (X_n, d) \to (X_n, \theta_n)\text{ is an } \eta\text{-quasisymmetry for each } n \in \mathbb{N}.
\]

By Proposition 3.21, there exist measures \(\mu_n\) on \(X_n\) for each \(n \in \mathbb{N}\), constant \(C_5 > 0\) such that

\[
C_5^{-1}r^{\delta}C_n(B) \leq \mu_n(B_{\theta_n}(x, r)) \leq C_5r^{\delta}C_n(B),
\]

for all \(x \in X_n, r < \text{diam}(X_n, \theta), B \in \mathcal{A}_S(B_d(x, s))\), where \(s\) is the largest number in \([0, 2\text{diam}(X_n, d)]\) such that \(B_d(x, s) \cap X_n \subset B_g(x, r)\) (as defined in (3.2)) and \(\mathcal{A}_S(B_d(x, s) \cap X_n)\) is as given in Definition 3.16 corresponding to the hyperbolic filling \(S^{(n)}\) of \(X_n\).
Next, normalize the metrics and measures by choosing a pair of sequences \( \beta_n, \gamma_n > 0 \) such that \( \hat{\theta}_n = \beta_n \theta_n, \hat{\mu}_n = \gamma_n \mu_n \) satisfy
\[
\text{diam}(B_d(x_0, 1), \hat{\theta}_n) = 1, \quad \hat{\mu}_n(B_d(o, 1)) = 1.
\] (4.42)
By (4.39) and (3.11), we note that \( B(x_0, 1) \subset X_n \) for all \( n \in \mathbb{N} \).

We choose \( p \in X \) such that \( d(x_0, p) = \frac{1}{2} \). Since \( \text{Id} : (X_n, d) \to (X_n, \hat{\theta}_n) \) is a \( \eta \)-quasisymmetry, by comparing the ratio of the diameter of the sets \( \{x_0, p\} \subset X_1 \) in the metrics \( d \) and \( \theta_n \) using (3.5), there exists \( C_{x_0, p} > 1 \) such that
\[
C_{x_0, p}^{-1} \leq \hat{\theta}_n(o, p) \leq C_{x_0, p}^{-1}, \text{ for all } n \in \mathbb{N}.
\]
We estimate \( \hat{\theta}_n(x, y)/\hat{\theta}_n(x_0, p) \) by writing it as \( \frac{\theta_n(x, y)}{\theta_n(x_0, x)} \frac{\theta_n(x_0, x)}{\theta_n(x_0, p)} \) and using \( \eta \)-quasisymmetry to estimate each of the factors and their reciprocals. This yields the following estimate; for any \( x, y \in X_n, n \in \mathbb{N} \), there exists \( C_{x,y} > 1 (C_{x,y} \text{ depends only on } d(x_0, x), d(x, y) \text{ and } \eta) \) such that
\[
C_{x,y}^{-1} \leq \hat{\theta}_n(x, y) \leq C_{x,y}^{-1}, \text{ for all } n \in \mathbb{N} \text{ such that } \{x, y\} \subset X_n.
\] (4.43)
By a similar computation, for any \( (x, y) \in X \times X \) and for any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that for any \( (x', y') \in X \times X \) with \( d(x, x') \lor d(y, y') < \delta \), we have
\[
\left| \hat{\theta}_n(x, y) - \hat{\theta}_n(x', y') \right| \leq \hat{\theta}_n(x, x') + \hat{\theta}_n(y, y') < \varepsilon,
\] (4.44)
By (4.44), (4.43) and Arzela-Ascoli theorem on the product space \( X_n \times X_n \) equipped with the product metric \( d_{ac}((x, y), (x', y')) = d(x, x') \lor d(y, y') \), the sequence of functions \( \hat{\theta}_m, m \geq n \), has a subsequence that converges uniformly to a metric \( \theta \) in \( X_n \). By a diagonalization argument, we obtain a subsequence of \( \{\hat{\theta}_n : n \in \mathbb{N}\} \), that converges uniformly in compact subsets of \( X \times X \). The limit metric \( \theta \in \mathcal{F}(X, d) \) and \( \text{Id} : (X, d) \to (X, \theta) \) is an \( \eta \)-quasisymmetry.

The measures \( \hat{\mu}_n \) constructed using Lemma 3.20 are uniformly doubling in the following sense: there exists \( C_D \geq 1 \) such that for all \( n \in \mathbb{N}, x \in X_n, 0 < r < \text{diam}(X_n, \hat{\theta}_n) \), we have
\[
\frac{\hat{\mu}_n(B_{\hat{\theta}_n}(x, 2r))}{\hat{\mu}_n(B_{\hat{\theta}_n}(x, r))} \leq C_D.
\]
By the argument in [LuS, Theorem 1], by a further diagonalization argument using weak*-compactness of \( \{\hat{\mu}_m : m \geq n\} \) on \( X_n \) for all \( n \in \mathbb{N} \), we obtain a measure \( \mu \) on \( X \). By (4.41), Proposition 4.9, there exist constants \( C_6 > 1, A_2 > 1 \) such that
\[
C_6^{-1} \frac{\mu(B_\theta(x, r))}{r^\beta} \leq \text{Cap}(B_\theta(x, r), B_\theta(x, 2r)^c) \leq C_4 \frac{\mu(B_\theta(x, r))}{r^\beta},
\] (4.45)
for any \( x \in X, r > 0 \). The remainder of the proof is exactly same as the compact case. Hence, we conclude that the walk dimension is two. \( \square \)
5 The attainment and Gaussian uniformization problems

In this section, we introduce the attainment problem for the conformal walk dimension and the Gaussian uniformization problem. Then we discuss partial progress towards them.

Let \((X, d, m, \mathcal{E}, \mathcal{F})\) be an MMD space that satisfies EHI. Recall that the Gaussian uniformization problem ask for a description of all metrics \(\theta \in \mathcal{J}(X, d)\) and measures \(\mu \in \mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F})\) such that the corresponding time-changed MMD space \((X, \theta, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu)\) satisfies PHI(2). For any \(\beta > 0\), we define

\[
G_\beta(X, d, m, \mathcal{E}, \mathcal{F}) = \left\{ \mu \in \mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F}) \mid \text{there exists } \theta \in \mathcal{J}(X, d) \text{ such that } (X, \theta, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu) \text{ satisfies PHI}(\beta) \right\}.
\]

(5.1)

We define the Gaussian admissible measures as

\[
\mathcal{G}(X, d, m, \mathcal{E}, \mathcal{F}) := G_2(X, d, m, \mathcal{E}, \mathcal{F}).
\]

(5.2)

By Theorem 2.10, we have

\[
G_\beta(X, d, m, \mathcal{E}, \mathcal{F}) = \emptyset \text{ for any } \beta < 2, \text{ and } G_\beta(X, d, m, \mathcal{E}, \mathcal{F}) \neq \emptyset \text{ for any } \beta > 2.
\]

This raises the following questions:

1. **Attainment problem**: Is \(\mathcal{G}(X, d, m, \mathcal{E}, \mathcal{F}) \neq \emptyset\)? Or equivalently, is the infimum in (1.3) attained?

2. **Gaussian uniformization problem**: Describe all measures in the set \(\mathcal{G}(X, d, m, \mathcal{E}, \mathcal{F})\).

By Proposition 2.11(a), the Gaussian admissible measures can be described as

\[
\mathcal{G}(X, d, m, \mathcal{E}, \mathcal{F}) = \left\{ \mu \in \mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F}) \mid d_{\text{int}}^\mu \in \mathcal{J}(X, d) \text{ and } (X, d_{\text{int}}^\mu, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu) \text{ satisfies PHI}(2) \right\},
\]

(5.3)

where \(d_{\text{int}}^\mu\) denotes the intrinsic metric of the Dirichlet form \((\mathcal{E}^\mu, \mathcal{F}^\mu)\) on \(L^2(X, \mu)\).

In this section, we prove Theorem 2.12 and discuss its consequences for the Gaussian uniformization problem. In particular, Theorem 2.12 shows that any two measures \(\mu_1, \mu_2 \in \mathcal{G}(X, d, m, \mathcal{E}, \mathcal{F})\) must be \(A_\infty\)-related in \((X, d)\) (provided such measures exist).

5.1 Consequences of PHI(2)

We begin with the proof of Proposition 2.11 which is essentially contained in [KM, §4]. This states that any measure on \(\mathcal{G}(X, d, m, \mathcal{E}, \mathcal{F})\) must be a minimal energy dominant measure and that the metric must be bi-Lipschitz equivalent to the intrinsic metric corresponding to the time-changed MMD space.

**Proof of Proposition 2.11.** By Theorem 4.4, we have all of the equivalent conditions in [GHL15, Theorem 1.2].
(a) We use [Mur20, Theorem 1.6 and Remark 1.7(a)] and [KM, Proposition 4.8] to obtain (a).

(b) This follows from [KM, Propositions 4.5 and 4.7].

\[\square\]

**Definition 5.1.** Let \((X, d)\) be a metric space and let \(u\) be a Borel measurable function. We define the **pointwise Lipschitz constant** as

\[
\text{Lip} u(x) := \limsup_{y \to x} \frac{|u(x) - u(y)|}{d(x, y)},
\]

and \(\text{Lip}(X)\) denotes the collection of all measurable functions \(u\) with

\[
\|u\|_{\text{Lip}(X)} := \sup_{x, y \in X, x \neq y} \frac{|u(x) - u(y)|}{d(x, y)} < \infty.
\]

When it is necessary, we also write \(\text{Lip}\) as \(\text{Lip}_d\) to specify the metric \(d\).

We recall the notion of upper gradient and its variants. We refer the reader to [HKST, Hei] for a comprehensive account.

**Definition 5.2.** Let \((X, d, m)\) be a metric measure space and let \(u : X \to \mathbb{R}\) be a Borel measurable function. A non-negative Borel measurable function \(g\) is called a **upper gradient**\(^2\) if

\[
|u(x) - u(y)| \leq \int_{\gamma} g \, ds,
\]

for every rectifiable curve \(\gamma\). A non-negative Borel measurable function \(g\) is called a **\(p\)-weak upper gradient of** \(u\) with \(p \in [1, \infty)\) if

\[
|u(x) - u(y)| \leq \int_{\gamma} g \, ds,
\]

for all \(\gamma \in \Gamma_{\text{rect}} \setminus \Gamma_0\), where \(x\) and \(y\) are the endpoints of \(\gamma\), \(\Gamma_{\text{rect}}\) denotes the collection of non-constant compact rectifiable curves and \(\Gamma_0\) has \(p\)-modulus zero in the sense that

\[
\inf \left\{ \|\rho\|_{L^p(X)}^p : \rho \text{ is non-negative, Borel measurable, } \int_{\gamma} \rho \, ds \geq 1 \text{ for all } \gamma \in \Gamma_0 \right\} = 0.
\]

We denote \(N^{1,p}(X)\) the collection of functions \(u \in L^p(X)\) that have a \(p\)-weak upper gradient \(g \in L^p(X)\), and define \(\|u\|_{N^{1,p}(X)} = \|u\|_{L^p(X)} + \inf_g \|g\|_{L^p(X)}\), where \(g\) is taken over all \(p\)-weak upper gradients of \(u\). We denote by \(N^{1,p}_{\text{loc}}(X)\) the class of functions \(u \in L^p_{\text{loc}}(X)\)

\(^2\)this notion is called **very weak gradient** in [HK]. Our terminology is borrowed from [HKST].
that have a $p$-weak upper gradient that belongs to $L^p(B)$ for each ball $B$. If necessary, we denote the spaces $N^{1,p}(X)$ and $N^{1,p}_{\text{loc}}(X)$ by $N^{1,p}(X, d, m)$ and $N^{1,p}_{\text{loc}}(X, d, m)$ respectively.

We say that $(X, d, m)$ is said to support a $(1, p)$-Poincaré inequality with $p \in [1, \infty)$ if there exist constants $K \geq 1, C > 0$ such that for all $u \in \text{Lip}(X), x \in X$ and $r > 0,$

$$\int_{B(x, r)} |u - u_{B(x, r)}| \, dm \leq C r \left[ \int_{B(x, Kr)} |(\text{Lip}(u))^p| \, dm \right]^{1/p},$$

where $\int_A f \, dm$ denotes $\frac{1}{m(A)} \int_A f \, dm$ and $u_{B(x, r)} = \frac{1}{m(B(x, r))} \int_{B(x, r)} u \, dm$. It is known that $(X, d, m)$ supports a $(1, p)$-Poincaré inequality if and only if there exist constants $K \geq 1, C > 0$ such that for every function $u$ that is integrable on balls and for any upper gradient $g$ of $u$ in $X$, $x \in X$ and $r > 0,$

$$\int_{B(x, r)} |u - u_{B(x, r)}| \, dm \leq C r \left[ \int_{B(x, Kr)} |g|^p \, dm \right]^{1/p},$$

where $u_{B(x, r)}$ is as above [HKST, Theorem 8.4.2].

We need the following self-improvement of Poincaré inequality.

**Proposition 5.3.** Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space that satisfies PHI(2). Then we have

(a) (Cf. [KZ12, Theorem 2.2]) $\mathcal{F} = N^{1,2}(X)$ with equivalent norms, $\text{Lip}(X) \cap C_c(X)$ is dense in $\mathcal{F}$ and $\mathcal{F}_{\text{loc}} = N^{1,2}_{\text{loc}}(X)$.

(b) (Cf. [KZ08, Theorem 1.0.1]) $(X, d, m)$ satisfies $(1, p)$-Poincaré inequality for some $p \in [1, 2)$.

**Proof.** (i) Let $d_{\text{int}}$ denote the intrinsic metric corresponding to the MMD space PHI(2). Since (VD) and PI(2) are preserved under a bi-Lipschitz change of metric (Cf. [HKST, Lemma 8.3.18]), by Proposition 2.11(a), the MMD space $(X, d_{\text{int}}, m, \mathcal{E}, \mathcal{F})$ also satisfies (VD) and PI(2). Therefore by [KZ12, Theorem 2.2], $\mathcal{F} = N^{1,2}(X, d_{\text{int}}, m)$ with equivalent norms and $\text{Lip}_{d_{\text{int}}}(X) \cap C_c(X)$ is dense in $\mathcal{F}$ and $\mathcal{F}_{\text{loc}} = N^{1,2}_{\text{loc}}(X, d_{\text{int}}, m)$. Since $d$ and $d_I$ are bi-Lipschitz equivalent, $\text{Lip}_{d_{\text{int}}}(X) = \text{Lip}(X)$ and $\text{Lip}_d(u)$ is comparable to $\text{Lip}_{d_{\text{int}}}(u)$; that is there exist constants $C > 0$ such that

$$C^{-1} \text{Lip}_d(u)(x) \leq \text{Lip}_{d_{\text{int}}}(u)(x) \leq C \text{Lip}_d(u)(x) \quad \text{for all } x \in X, u \in \text{Lip}_d(X). \quad (5.4)$$

(ii) By (i), [KZ08, Proposition 2.1] and [HKST, Lemma 8.3.18], $(X, d, m)$ satisfies the $(1,2)$-Poincaré inequality. By the self-improving property of [KZ08, Theorem 1.0.1], $(X, d, m)$ satisfies $(1, p)$-Poincaré inequality for some $p \in [1, 2)$.
5.2 $A_\infty$-weights and the Gaussian uniformization problem

**Definition 5.4 ($A_\infty$ relation).** Let $(X, d, m)$ be a complete metric measure space such that $m$ is a doubling measure. Let $m'$ be another doubling Borel measure on $X$. Then $m'$ is said to be $A_\infty$-related to $m$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$m(E) < \delta m(B) \quad \text{implies} \quad m'(E) < \varepsilon m'(B)$$

whenever $E$ is a measurable subset of a ball $B$. Evidently, if $m'$ is $A_\infty$-related to $m$, then $m'$ is absolutely continuous with respect to $m$, so that $dm' = wdm$ for some nonnegative locally integrable weight function $w$. It turns out that being $A_\infty$-related is a symmetric relation among doubling measures; that is, if $m'$ is $A_\infty$-related to $m$, then $m$ is $A_\infty$-related to $m'$ [ST, Chapter I].

Consider the following reverse Hölder inequality: There is a locally $m$-integrable function $w$ in $X$ together with constants $C \geq 1$ and $p > 1$ such that $dm' = wdm$ and

$$\left(\int_B w^p dm\right)^{1/p} \leq C \int_B w dm \quad (5.5)$$

whenever $B$ is a ball in $(X, d)$. It is well known that a doubling measure $m'$ is $A_\infty$-related to $m$ if and only if the reverse Hölder inequality (5.5) is satisfied.

The $A_\infty$ relation among doubling measures is preserved under quasisymmetric change of metric as we show below.

**Lemma 5.5.** Let $d_1, d_2$ two quasisymmetric metrics on $X$ such that the metrics $d_1, d_2$ are uniformly perfect. Let $m_1, m_2$ be two doubling Borel measures with respect to $d_1$ such that $m_1$ and $m_2$ are $A_\infty$ with respect to the metric $d_1$. Then $m_1$ and $m_2$ are $A_\infty$ related with respect to $d_2$.

**Proof.** Let $B_i(x, r)$ denote the open ball in metric $d_i$ for $i = 1, 2$. By [MT10, Lemma 1.2.18], there exists $C > 0$ such that the following holds: for each $x \in X, r > 0$ and $i \in \{1, 2\}$, there exists $s > 0$ such that

$$B_{3^{-i}}(x, C^{-1}s) \subseteq B_i(x, r) \subseteq B_{3^{-i}}(x, Cs), \quad \text{for all } x \in X, r > 0. \quad (5.6)$$

Note that since $m_1$ and $m_2$ are doubling with respect to $d_1$, they are also doubling on $(X, d_2)$. Therefore, there exists $C_1 > 0$ such that

$$\frac{m_i(B_j(x, Cr))}{m_i(B_j(x, r))} \leq C_1, \quad \text{for all } x \in X, r > 0, \text{ and } i, j \in \{1, 2\}. \quad (5.7)$$

Since $m_1$ and $m_2$ are $A_\infty$-related in $(X, d_1)$, we have $m_2 \ll m_1$, $dm_2 = w dm_1$, where $w \geq 0$ is a Borel measurable function that satisfies the following reverse Hölder inequality: there exists $C_R \geq 1, p > 1$ such that

$$\left(\int_{B_1(x, r)} w^p dm_1\right)^{1/p} \leq C_R \int_{B_1(x, r)} w dm_1, \quad \text{for all } x \in X, r > 0. \quad (5.8)$$
For all $x \in X, r > 0$, we estimate
\[
\left( \int_{B_2(x,r)} w^p \, dm_1 \right)^{1/p} \leq \left( \int_{B_1(x,Cs)} w^p \, dm_1 \right)^{1/p} \left( \frac{m_1(B_1(x,Cs))}{m_1(B_1(x,C^{-1}s))} \right)^{1/p} \quad \text{(by (5.6))}
\]
\[
\leq C_R C_1^{2/p} \int_{B_1(x,Cs)} w \, dm_1 \quad \text{(by (5.8) and (5.7))}
\]
\[
= C_R C_1^{2/p} \frac{m_2(B_1(x,Cs))}{m_1(B_1(x,Cs))} \quad \text{(by $dm_2 = w \, dm_1$)}
\]
\[
\leq C_R C_1^{2/p} \frac{m_2(B_1(x,C^{-1}s))}{m_1(B_2(x,r))} \quad \text{(by (5.6))}
\]
\[
\leq C_R C_1^{4/p} \frac{m_2(B_1(x,C^{-1}s))}{m_1(B_2(x,r))} \quad \text{(by (5.7))}
\]
\[
\leq C_R C_1^{4/p} \frac{m_2(B_2(x,r))}{m_1(B_2(x,r))} \quad \text{(by (5.6))}
\]
\[
\leq C_R C_1^{4/p} \int_{B_2(x,r)} w \, dm_1, \quad \text{(since $dm_2 = w \, dm_1$)}
\]

\[\square\]

Let $f : (X_1,d_1) \to (X_2,d_2)$ be a homeomorphism between two metric spaces. For all $x \in X, r > 0$, we define
\[
L_f(x, r) = \sup \{d_2(f(x), f(y)) : d_1(x,y) \leq r\}, \quad L_f(x) = \limsup_{r \to 0} \frac{L_f(x, r)}{r}. \quad (5.9)
\]
For $\varepsilon > 0$, we define
\[
L_f^\varepsilon(x) = \sup_{0 < r \leq \varepsilon} \frac{L_f(x, r)}{r}. \quad (5.10)
\]
Clearly, $L_f$ decreases as $\varepsilon$ decreases and
\[
\lim_{\varepsilon \to 0} L_f^\varepsilon(x) = L_f(x), \quad \text{for all } x \in X.
\]

**Lemma 5.6.** ([HK, Lemma 7.16]) Let $f : (X_1,d_1) \to (X_2,d_2)$ be a $\eta$-quasisymmetry. Let $x_0 \in X$ and $0 < R < \text{diam}(X,d_1)$, there is a constant $C \geq 1$ (that depends only on $\eta$) such that for all $\varepsilon > 0$, the function $CL_f^\varepsilon$ is an upper gradient of the function $u(x) = d_2(f(x), f(x_0))$ in $B(x_0, R)$.

We introduce the notion of $C$-approximation to compare balls in different metrics.

**Definition 5.7.** Let $d_1$ and $d_2$ be two metrics on $X$ such that the identity map $\text{Id} : (X,d_1) \to (X,d_2)$ is a $\eta$-quasisymmetry. Let $C \geq 1$ be a constant. We say that a ball $B_{d_2}(x_2, r_2)$ is a $C$-approximation of $B_{d_1}(x_1, r_1)$ if
\[
d_1(x_1, x_2) \leq Cr_1, \quad d_2(x_1, x_2) \leq Cr_2
\]
\[
B_{d_1}(x_2, C^{-1}r_1) \subset B_{d_2}(x_2, r_2) \subset B_{d_1}(x_2, Cr_1),
\]
\[
B_{d_2}(x_1, C^{-1}r_2) \subset B_{d_1}(x_1, r_1) \subset B_{d_2}(x_1, Cr_2)
\]
By the same argument as Proposition 4.9, we obtain the following comparison of capacities.

**Lemma 5.8.** Let \((X, d_i, m_i, \mathcal{E}, \mathcal{F}_i), i = 1, 2,\) be two geodesic, MMD spaces that satisfy EHI such that the identity map \(\text{Id} : (X, d_1) \to (X, d_2)\) is a quasisymmetry. Let \(B_i(x, r)\) denote a open ball of radius \(r\) and center \(x,\) for \(i = 1, 2.\) Let \(C_1 \geq 1\) and \(A_1, A_2 > 1.\) There exists \(C_2, A_3 > 1\) such that

\[
\text{Cap}(B_1(x_1, r_1), B_1(x_1, A_1 r_1)^c) \leq C_2 \text{Cap}(B_2(x_2, r_2), B_2(x_2, A_2 r_2)^c)
\]

for all balls \(B_1(x_1, r_1)\) and \(B_2(x_2, r_2)\) such that \(r_1 < \text{diam}(X, d_1)/A_3, r_2 < \text{diam}(X, d_2)/A_3\) and that \(B_1(x_1, r_1)\) is a \(C_1\)-approximation of \(B_2(x_2, r_2).\)

The following is an analogue of [HK, Lemma 7.19]

**Lemma 5.9.** Let \((X, d_i, m_i, \mathcal{E}, \mathcal{F}_e \cap L^2(m_i)), i = 1, 2,\) be two MMD spaces that satisfy PHI(2). Furthermore, these MMD spaces are time changes of each other with full quasi-support. Let the identity map \(f : (X, d_1) \to (X, d_2)\) be a \(\eta\)-quasisymmetry. Then the function \(L^*_f\) defined in (5.10) is in weak \(L^2\) for any \(\varepsilon < R/10\) and for any ball \(B_{d_i}(x_0, R), R \leq \text{diam}(X, d_1).\) Furthermore, there exists \(C \geq 1\) such that \(L^*_f\) satisfies the estimate

\[
m_1 \left\{ x \in B_{d_i}(x_0, R) : L^*_f(x) > t \right\} \leq Ct^{-2} m_2 (B_{d_i}(x_0, R)),
\]

for all \(t > 0, 0 < R < \text{diam}(X, d_1), x_0 \in X.\) Here \(C \geq 1\) depends only on \(\eta\) and the constants associated with the MMD spaces \((X, d_i, m_i, \mathcal{E}, \mathcal{F}_e \cap L^2(m_i)), i = 1, 2.\)

**Proof.** Let \(E_t\) denote the set

\[
E_t := \left\{ x \in B_{d_i}(x_0, R) : L^*_f(x) > t \right\}.
\]

Then by the 5B-covering lemma [Hei, Theorem 1.2] there exists a countable collection of disjoint balls \(B_i = B_{d_i}(x_i, r_i), i \in I\) such that \(0 < r_i \leq \varepsilon,\)

\[
\frac{L_f(x_i, r_i)}{r_i} > t
\]

and

\[
E_t \subset \cup_i 5B_i \subset 2B.
\]

Note that the metrics \(d_1, d_2\) are uniformly perfect by Proposition 2.11(a). Define

\[
B'_i := B_{d_2}(x_i, L_f(x_i, r_i)/\eta(1)).
\]

Roughly speaking, the balls \(B'_i\) in \(d_2\)-metric approximate the balls \(B_i\) in the \(d_1\)-metric for each \(i \in I.\) More precisely, since \(f\) is a \(\eta\)-quasisymmetry and \(d_1, d_2\) are uniformly perfect, there exists \(C \geq 1\) such that \(B_i\) is a \(C\)-approximation of \(B'_i\) for all \(i \in I.\) In particular,

\[
C^{-1}B'_i \subset B_i \subset CB'_i, \quad C^{-1}B_i \subset B'_i \subset CB_i, \quad \text{for all } i \in I.
\]
Since \((X, d_i, m_i, \mathcal{E}, \mathcal{F}_e \cap L^2(m_i))\), \(i = 1, 2\) satisfies \(
abla\text{PHI}(2)\), there exists \(A_1, A_2 > 1\) such that

\[
\text{Cap}(B, (A_1 B)_c) \leq \frac{\nu_i^2}{m_i(B)}, \quad \text{Cap}(B_i', (A_2 B_i')_c) \approx \frac{L_f(x_i, r_i)^2}{m_2(B_i')} \quad \text{for all } i \in I. \tag{5.14}
\]

Furthermore, since by (5.13) and Lemma 5.8, we have

\[
\text{Cap}(B, (A_1 B)_c) \leq \text{Cap}(B_i', (A_2 B_i')_c) \quad \text{for all } i \in I. \tag{5.15}
\]

We combine the above estimates, to obtain (5.11) as follows:

\[
m_1(E_i) \leq \sum_i m_1(5B_i) \lesssim \sum_i m(B_i) \quad \text{(since } m_1 \text{ is a doubling)}
\]

\[
\lesssim \sum_i \nu_i^2 \text{Cap}(B_i, (A_1 B_i)_c) \quad \text{(by (5.14))}
\]

\[
\lesssim t^{-2} \sum_i L_f(x_i, r_i)^2 \text{Cap}(B_i, (A_1 B_i)_c) \quad \text{(by (5.12))}
\]

\[
\lesssim t^{-2} \sum_i L_f(x_i, r_i)^2 \text{Cap}(B_i', (A_2 B_i')_c) \quad \text{(by (5.15))}
\]

\[
\lesssim t^{-2} \sum_i m_2(B_i') \quad \text{(by (5.14))}
\]

\[
\lesssim t^{-2} \sum_i m_2(C^{-1} B_i') \quad \text{(by (VD) for } (X, d_2, m_2))
\]

\[
\lesssim t^{-2} \sum_i m_2(B_i) \quad \text{(since } C^{-1} B_i' \subset B_i)
\]

\[
\lesssim t^{-2} \sum_i m_2(B_{d_i}(x_0, 2R)) \quad \text{(since } B_i \text{'s are disjoint and } \cup_i B_i \subset B_{d_i}(x_0, 2R))
\]

\[
\lesssim t^{-2} m_2(B_{d_i}(x_0, R)) \quad \text{(since } (X, d_1, m_2) \text{ is doubling).}
\]

The dependence of the constant \(C\) in (5.11) follows from the above argument. \(\square\)

**Corollary 5.10.** ([HK, Corollary 7.21]) Let \((X, d_i, m_i, \mathcal{E}, \mathcal{F}_e \cap L^2(m_i))\), \(i = 1, 2\) be two MMD spaces that satisfy \(\nabla\text{PHI}(2)\). Furthermore, these MMD spaces are time changes of each other. Let the identity map \(f : (X, d_1) \to (X, d_2)\) be a \(\eta\)-quasisymmetry. Then the function \(L_f^s\) defined in (5.10). Let \(L_f^s\) denote the function defined in (5.10). For all \(s \in [1, 2]\) and \(x_0 \in X, 0 < \varepsilon < R/10, R \leq \text{diam}(X, d_1)\), the function \(L_f^s\) is in \(L^s(B_{d_i}(x_0, R), m_1)\) with

\[
\left( \int_{B_{d_i}(x_0, R)} |L_f^s|^s \, dm_1 \right)^{1/s} \leq C m_1(B(x_0, R))^{(2-s)/(2s)} m_2(B(x_0, R))^{1/s},
\]

where \(C\) only depends only on \(s, \eta\) and the constants associated with the two MMD spaces. By letting \(\varepsilon \downarrow 0\), a similar statement is true for \(L_f\).
Proof of Theorem 2.12. By Proposition 2.11(b), both $m_1$ and $m_2$ are minimal energy dominant measures. Therefore, $m_1$ and $m_2$ are mutually absolute continuous. By Proposition 2.11(a), both $d_1$ and $d_2$ are bi-Lipschitz equivalent to intrinsic metrics, and therefore by Lemma 5.5, we may assume that $d_1$ and $d_2$ are intrinsic metrics with respect to the symmetric measures $m_1$ and $m_2$ respectively.

Let $f : (X, d_1) \to (X, d_2)$ denote the identity map that is a $\eta$-quasisymmetry. Then by the Lebesgue-Radon-Nikodym theorem, the volume derivative
\[
\mu_f(x) = \lim_{r \downarrow 0} \frac{m_2(B_d(x, r))}{m_1(B_d(x, r))}
\]exists and is finite for $m_1$-almost every $x \in X$. Since $m_2 \ll m_1$, we have $dm_2 = \mu_f dm_1$; that is $m_2(E) = \int_E \mu_f dm_1$ for all measurable sets $E$.

Since $(X, d_i, m_i, \mathcal{E}, \mathcal{F} \cap L^2(m_i))$ satisfies PHI(2) for $i = 1, 2$, there exists constants $A_1, A_2, C_1, C_2$ such that
\[
\text{Cap}(B_d(x, r), B_d(x, A_i r)^c) \asymp r^2 \frac{m_1(B_d(x, r))}{m_i(B_d(x, r))}, \text{ for all } x \in X, r < \text{diam}(X, d_i)/C_i, i = 1, 2.
\]Similar to (5.13), there exists $C \geq 1$ such that for all $r < \text{diam}(X, d_1)$, $x \in X$, $B_d(x, r)$ is a $C$-approximation of $B_{d_2}(x, Lf(x, r))$. That is, for all $r < \text{diam}(X, d_1)$, $x \in X$, $B_d(x, r)$,
\[
B_{d_2}(x, C^{-1}L_f(x, r)) \subset B_{d_1}(x, r) \subset B_{d_2}(x, CL_f(x, r)),
\]
\[
B_{d_1}(x, C^{-1}r) \subset B_{d_2}(x, L_f(x, r)) \subset B_{d_1}(x, Cr).
\]By (5.18), $\eta$-quasisymmetry of $f$, and the same argument as Proposition 4.9, there exists $C_3 > 1$ such that
\[
\text{Cap}(B_{d_1}(x, r), B_{d_1}(x, A_1 r)^c) \asymp \text{Cap}(B_{d_2}(x, L_f(x, r)), B_{d_2}(x, A_2 L_f(x, r))^c),
\]for all $x \in X, r < \text{diam}(X, d_1)/C_3$. Combining (5.17) and (5.19),
\[
\frac{L_f(x, r)^2}{r^2} \asymp \frac{m_2(B_{d_2}(x, L_f(x, r)))}{m_1(B_d(x, r))} \asymp \frac{m_2(B_{d_1}(x, r))}{m_1(B_d(x, r))}, \text{ for all } x \in X, r < \text{diam}(X, d_1).
\]Therefore
\[
\mu_f(x) \asymp L_f(x)^2 \text{ for almost every } x \in X.
\]Let $p \in [1, 2)$ be the constant in Proposition 5.3(b) so that $(X, d_1, m_1)$ satisfies $(1, p)$-Poincaré inequality. We shall show that $L_f$ satisfies the reverse Hölder inequality
\[
\left(\int_{B_{d_1}(x_0, r)} L_f^2 \ d m_1\right)^{1/2} \leq C \left(\int_{B_{d_1}(x_0, r)} L_f^p \ d m_1\right)^{1/p}, \text{ for all } x_0 \in X, r < \text{diam}(X, d_1).
\]
Then by Gehring’s lemma [HK, Lemma 7.3], Hölder inequality and (5.22), we obtain the following reverse Hölder inequality for the function $\mu_f$: there exists $\varepsilon > 0$ such that
\[
\left( \frac{\int_{B_{d_1}(x_0,r)} \mu_f^{1+\varepsilon} \, dm_1}{\int_{B_{d_1}(x_0,r)} \mu_f \, dm_1} \right)^{1/(1+\varepsilon)} \leq C \int_{B_{d_1}(x_0,r)} \mu_f \, dm_1, \text{ for all } x_0 \in X, \ r < \text{diam}(X, d_1).
\]

By the equivalence between reverse Hölder inequality and $A_\infty$ relation as explained in Definition 5.4, it suffices to show (5.22).

Since $(X, d_1, m_1)$ satisfies the $(1, p)$ Poincaré inequality, by Lemma 5.6 $CL_\varepsilon$ is an upper gradient of $u(x) = d_2(x, x_0)$ in $B_{d_1}(x_0, r)$. Therefore by the Poincaré inequality we have
\[
\int_{B_{d_1}(x_0,K_{-1}r)} \left| u - u_{B_{d_1}(x_0,K_{-1}r)} \right| \, dm_1 \lesssim r \left( \int_{B_{d_1}(x_0,r)} (L_\varepsilon f)^p \, dm_1 \right)^{1/p}.
\]

We let $\varepsilon \downarrow 0$ and use Corollary 5.10, and dominated convergence theorem, to obtain
\[
\int_{B_{d_1}(x_0,K_{-1}r)} \left| u - u_{B_{d_1}(x_0,K_{-1}r)} \right| \, dm_1 \lesssim r \left( \int_{B_{d_1}(x_0,r)} |L_f|^p \, dm_1 \right)^{1/p}. \tag{5.23}
\]

By the uniform perfectness of $(X, d_1)$ and the volume doubling property, there exists $K_1$ such that $m_1 \left( B(x_0, K_{-1}r) \setminus B(x_0, K_{1}^{-1}K_{-1}r) \right) \gtrsim m_1(B(x_0, r))$. Using the quasisymmetry of $f$, we obtain
\[
\begin{align*}
    u_{B_{d_1}(x_0,K_{-1}r)} = & \int_{B_{d_1}(x_0,K_{-1}r)} d_2(x, x_0) \, m_1(dx) \\
    \geq & \frac{1}{m_1(B_{d_1}(x_0,r))} \int_{B(x_0,K_{-1}r) \setminus B(x_0,K_{1}^{-1}K_{-1}r)} d_2(x, x_0) \, m_1(dx) \\
    \gtrsim & L_f(x_0, r) \frac{m_1 \left( B(x_0, K_{-1}r) \setminus B(x_0, K_{1}^{-1}K_{-1}r) \right)}{m_1(B(x_0, r))} \\
    \geq & C_1^{-1} L_f(x_0, r),
\end{align*}
\]

because
\[
    L_f(x_0, r) \lesssim d_2(x, x_0)
\]

by quasisymmetry of $\eta$ and the uniform perfectness of $(X, d_1)$. For sufficiently small $\delta > 0$, we similarly have
\[
    u(x) = d_2(x, x_0) \leq \eta(\delta K_2) L_f(x_0, r) \leq (2C_1)^{-1} L_f(x_0, r)
\]

for all $x \in B(x_0, \delta K_{-1}r)$. Consequently, using the above estimates and the volume doubling property we obtain
\[
\int_{B_{d_1}(x_0,K_{-1}r)} \left| u - u_{B_{d_1}(x_0,K_{-1}r)} \right| \, dm_1 \gtrsim \int_{B_{d_1}(x_0,\delta K_{-1}r)} \left| u - u_{B_{d_1}(x_0,K_{-1}r)} \right| \, dm_1 \gtrsim L_f(x_0, r). \tag{5.24}
\]
Combining the above estimate
\[
\left( \int_{B_{d_1}(x_0,r)} L_f^2 \, dm_1 \right)^{1/2} \lesssim \left( \int_{B_{d_1}(x_0,r)} \mu_f \, dm_1 \right)^{1/2} \quad \text{(by (5.21))}
\]
\[
\lesssim \left( \frac{m_2(B_{d_2}(x_0,r))}{m_2(B_{d_1}(x_0,r))} \right)^{1/2} \quad \text{(since } dm_2 = \mu_f \, dm_1)\]
\[
\lesssim \frac{L_f(x_0,r)}{r} \quad \text{(by (5.20))}
\]
\[
\lesssim \left( \int_{B_{d_1}(x_0,r)} |L_f|^p \, dm_1 \right)^{1/p} \quad \text{(by (5.24) and (5.23)).}
\]

This completes the proof of (5.22), and therefore Theorem 2.12. □

Let \((X,d,m,\mathcal{E},\mathcal{F})\) be a MMD space that satisfies EHI, where \((X,d)\) is a doubling metric space. If \(\mu \in \mathcal{G}(X,d,m,\mathcal{E},\mathcal{F})\), then by Theorem 2.12
\[
\mathcal{G}(X,d,m,\mathcal{E},\mathcal{F}) \subseteq \{ \tilde{\mu} : \tilde{\mu} \text{ is } A_\infty\text{-related to } \mu \}. \quad (5.25)
\]

One might ask if the inclusion in (5.25) is strict. For the Brownian motion on \(\mathbb{R}^n\), the above inclusion is strict if and only if \(n \geq 2\) (see Theorem 5.17 and Example 5.13). We need the definition of a maximal semi-metric.

**Definition 5.11.** A function \(r : X \times X \to [0, \infty)\) is said to be semi-metric, if it satisfies all the properties of a metric except possibly the property that \(r(x,y) = 0\) implies \(x = y\).

Let \(h : X \times X \to [0, \infty)\) be an arbitrary function. Then there exists a unique maximal semi-metric \(d_h : X \times X \to [0, \infty)\) such that \(d_h(x,y) \leq h(x,y)\) for all \(x,y \in X\) [BBI, Lemma 3.1.23]. We say that \(d_h\) is the maximal semi-metric induced by \(h\). Equivalently, \(d_h\) can be defined as follows. Let \(\tilde{h}(x,y) = \min(h(x,y), h(y,x))\). Then
\[
d_h(x,y) = \inf \left\{ \sum_{i=0}^{N-1} \tilde{h}(x_i, x_{i+1}) : N \in \mathbb{N}, x_0 = x, x_N = y \right\}. \quad (5.26)
\]

We provide a necessary condition for a metric to be in \(\mathcal{G}(X,d,m,\mathcal{E},\mathcal{F})\). Using the below necessary condition, we obtain examples for which the inclusion (5.25) is not strict.

**Lemma 5.12.** Let \((X,d,m,\mathcal{E},\mathcal{F})\) satisfy PHI(\(\gamma\)) for some \(\gamma \geq 2\). Let \(\mu \in \mathcal{G}(X,d,m,\mathcal{E},\mathcal{F})\). Define
\[
h(x,y) = \sqrt{\frac{\mu(B_d(x,d(x,y))) d(x,y)^\gamma}{m(B_d(x,d(x,y)))}}, \quad \text{for any } x,y \in X \text{ with } x \neq y, \quad (5.27)
\]
and \(h(x,x) = 0\) for any \(x \in X\). Let \(d_h\) denote the maximal semi-metric induced by \(h\). Then there exist \(C > 0\) such that
\[
h(x,y) \leq C d_h(x,y) \quad \text{for all } x,y \in X.
\]
Proof. Let \( \theta \in \mathcal{J}(X,d) \) such that the MMD space \((X,\theta,\mathcal{E}^\mu,\mathcal{F}^\mu)\) satisfies PHI(2). It suffices to show the existence of \( C_1 > 0 \) such that

\[
C_1^{-1} \theta(x,y) \leq h(x,y) \leq C_1 \theta(x,y) \quad \text{for all } x, y \in X. \tag{5.28}
\]

In particular, (5.28) implies a similar inequality as eqref:ns1 with \( h(x,y) \) replaced with \( d_h(x,y) \), which immediately implies that \( h \) is comparable to \( d_h \).

By Theorem 4.4, \( m \) and \( \mu \) satisfy VD on \((X,d)\) and \((X,\theta)\). Using Proposition 3.2 and VD, there exists \( C_2 > 0 \) such that

\[
C_2^{-1} \mu(B_\theta(x,\theta(x,y))) \leq \mu(B_d(x,d(x,y))) \leq C_2 \mu(B_\theta(x,\theta(x,y))) \tag{5.29}
\]

for all \( x, y \in X \), where \( x \neq y \). By an argument similar to the proof of Proposition 4.9, using Lemma 4.8 and Proposition 3.2, there exists \( C_3, A > 0 \) such that

\[
C_3^{-1} \leq \frac{\text{Cap}(B_\theta(x,\theta(x,y)), B_\theta(x,2\theta(x,y))^e)}{\text{Cap}(B_d(x,d(x,y)), B_d(x,2d(x,y))^e)} \leq C_3, \tag{5.30}
\]

for any pair \( x, y \in X \) such that \( 0 < \theta(x,y) < \text{diam}(X,\theta)/A \). By Theorem 4.4, Lemma 4.8 and Proposition 3.2 and increasing \( A \) if necessary, there exist \( C_4 > 0 \) such that

\[
C_4^{-1} \frac{\mu(B_\theta(x,\theta(x,y)))}{\theta(x,y)^2} \leq \frac{\text{Cap}(B_\theta(x,\theta(x,y)), B_\theta(x,2\theta(x,y))^e)}{\text{Cap}(B_d(x,d(x,y)), B_d(x,2d(x,y))^e)} \leq C_4 \frac{\mu(B_\theta(x,\theta(x,y)))}{\theta(x,y)^2}, \tag{5.31}
\]

and

\[
C_4^{-1} \frac{\mu(B_\theta(x,\theta(x,y)))}{\theta(x,y)^\gamma} \leq \frac{\text{Cap}(B_\theta(x,\theta(x,y)), B_\theta(x,2\theta(x,y))^e)}{\text{Cap}(B_d(x,d(x,y)), B_d(x,2d(x,y))^e)} \leq C_4 \frac{m(B_d(x,d(x,y)))}{d(x,y)\gamma}, \tag{5.32}
\]

for any pair \( x, y \in X \) such that \( 0 < \theta(x,y) < \text{diam}(X,\theta)/A \). Combining (5.29), (5.30), (5.31), and (5.32), there exists \( C_5 > 0 \) such that

\[
C_5^{-1} \theta(x,y) \leq h(x,y) \leq C_5 \theta(x,y) \quad \text{for all } x, y \in X \text{ such that } 0 \leq \theta(x,y) < \text{diam}(X,\theta)/A. \tag{5.33}
\]

Since \((X,\theta)\) is uniformly perfect, by replacing \( y \) with a closer point \( \tilde{y} \), and using (5.33), VD, we obtain (5.28).

Example 5.13. Let \( n \geq 2 \) and let \((X,d,m,\mathcal{E},\mathcal{F})\) denote the Dirichlet form corresponding to Brownian motion on \( \mathbb{R}^n \). If \( w(x) = |x|^{t} \), where \( t \in \mathbb{R}, x = (x_1,\ldots,x_n) \), then \( w \, dm \) is \( A_\infty \)-related to \( m \) if and only if \( t > -1 \) [Sem93, p. 222, Example (c)]. If \( t > 0, x = (0,\ldots,0), y = (0,\ldots,0,1) \) and \( h \) is as given in (5.27) with \( \gamma = 2 \), then using (5.26) it is easy to check that \( d_h(x,y) = 0 \). This can be seen by choosing equally spaced points \( x_0,\ldots,x_n \) on the straight line joining \( x \) and \( y \) and letting \( n \to \infty \) in (5.26). Therefore, by Lemma 5.12 we obtain

\[
w \, dm \in \{ \tilde{\mu} : \tilde{\mu} \text{ is } A_\infty \text{-related to } \mu \} \setminus \mathcal{G}(X,d,m,\mathcal{E},\mathcal{F}), \quad \text{for any } n \geq 2, t > 0.
\]

In other words, the inclusion in (5.25) is strict for the Brownian motion on \( \mathbb{R}^n, n \geq 2 \).
The above example and Lemma 5.12 illustrate that if a measure is too small in the neighborhood of a curve, then it will fail be in $\mathcal{G}(X, d, m, \mathcal{E}, \mathcal{F})$. As we will see, a similar (but more subtle) phenomenon happens in higher dimensional Sierpinski gaskets.

We recall the definition of strong $A_\infty$-weights on $\mathbb{R}^n$ introduced by David and Semmes in [DS] and show its relevance to the Gaussian uniformization problem for the Brownian motion on $\mathbb{R}^2$. The following definition is a slight reformulation of the one in [DS] and follows from [Sem93, Lemma 3.1].

**Definition 5.14.** Let $d, m$ denote the Euclidean metric and Lebesgue measure on $\mathbb{R}^n$ respectively. Let $\mu = wd m$ be $A_\infty$-related to $m$. Define

$$h(x, y) = (\mu(B_{x,y}))^{1/n},$$

where $B_{x,y}$ is the Euclidean ball with center $z = (x + y)/2$ and radius $d(x, y)/2$. Let $d_h$ denote the maximal metric We say that $\mu$ is strong $A_\infty$-related to $m$, if there exists $C > 0$ such that

$$d_h(x, y) \geq C^{-1}h(x, y), \quad \text{for all } x, y \in \mathbb{R}^n.$$

The following relates the Gaussian uniformization problem in $\mathbb{R}^2$ in terms of strong $A_\infty$-weights.

**Proposition 5.15.** Let $(X, d, m, \mathcal{E}, \mathcal{F})$ denote the MMD space corresponding to the Brownian motion on $\mathbb{R}^2$. Then

$$\mathcal{G}(X, d, m, \mathcal{E}, \mathcal{F}) = \{\tilde{\mu} : \tilde{\mu} \text{ is strong } A_\infty \text{-related to } \mu\}. \quad (5.34)$$

**Proof.** By Lemma 5.12 and Theorem 2.12, we have the inclusion

$$\mathcal{G}(X, d, m, \mathcal{E}, \mathcal{F}) \subseteq \{\tilde{\mu} : \tilde{\mu} \text{ is strong } A_\infty \text{-related to } \mu\}. \quad (5.35)$$

For the reverse inclusion, consider a measure $\mu = wdm$ that is $A_\infty$-related to $m$. Since $\mu$ is mutually absolutely continuous with respect to $m$, $\mu \in \mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F})$.

Let $\theta$ denote the metric $d_h$ in Definition 5.14. Since $\mu$ is $A_\infty$-related to $m$, $\mu$ satisfies VD on $(X, d)$. Since $\theta(x, y)$ is comparable to $\sqrt{\mu(B_{x,y})}$, where $B_{x,y} = B((x+y)/2, d(x, y)/2)$, by the VD and RVD for the measure $\mu$, we obtain that

$$\theta \in \mathcal{J}(X, d). \quad (5.36)$$

By (5.36), Proposition 3.2 and VD, there exists $C_1 > 0$ such that

$$C_1^{-1}r^2 \leq \mu(B_{\theta}(x, r)) \leq C_1r^2 \quad \text{for all } x \in X, r > 0. \quad (5.37)$$

By Lemma 4.8, Proposition 3.2, and (5.36), there exist $C_2 > 0$ such that

$$C_2^{-1} \leq \text{Cap}(B_{\theta}(x, r), B_{\theta}(x, 2r))^c \leq C_2 \quad \text{for all } x \in X, r > 0. \quad (5.38)$$
By [BM, Lemma 5.3], the time changed MMD space $(X, \theta, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu)$ satisfies EHI. Combining (5.36), (5.37), (5.38), (5.36), and using Theorem 4.4, we obtain that $\mu \in \mathcal{G}(X, d, m, \mathcal{E}, \mathcal{F})$. □

Proposition 5.15 along with known results on strong $A_\infty$ related measures lead to many further examples of measures in $\mathcal{G}(X, d, m, \mathcal{E}, \mathcal{F})$ for Brownian motion in $\mathbb{R}^2$. For instance, Bessel potentials can be used to construct strong $A_\infty$-measures [BHS, Theorem 3.1].

### 5.3 Gaussian uniformization problem on the real line

In this section, we answer the Gaussian uniformization problem for Brownian motion on $\mathbb{R}$. Let $(X, d, m, \mathcal{E}, \mathcal{F})$ denote an MMD space that satisfies the EHI. In particular, we show that the inclusion in (5.25) is an equality for the one-dimensional Brownian motion.

We need a lemma that characterizes $A_\infty$-related weights in terms of a reverse Hölder inequality.

**Lemma 5.16.** Let $(X, d)$ be a metric space and let $m$ be a doubling measure such that the function $r \mapsto m(B(x, r))$ increases continuously in $r$ for each $x$. Then the following are equivalent:

(a) $\mu = w \, dm$ is $A_\infty$ related to $m$.

(b) $w \not\equiv 0$ and there exists $C > 1$ such that the following reverse Hölder condition holds:

$$
\int_B w \, dm \leq C \left( \int_B \sqrt{w} \, dm \right)^2, \quad \text{whenever } B \text{ is a ball in } (X, d).
$$

**Proof.** (b) $\implies$ (a): By Gehring’s lemma [HK, Lemma 7.3], there exists $\varepsilon > 0, C_1 > 0$ such that

$$
\left( \int_B w^{1+\varepsilon} \, dm \right)^{1/(1+\varepsilon)} \leq C_1 \left( \int_B \sqrt{w} \, dm \right)^2 \leq \int_B w \, dm,
$$

for all balls $B$ in $(X, d)$. By [ST, Theorem 18 in Chapter I], $\mu$ is $A_\infty$-related to $m$.

(a) $\implies$ (b): By [ST, Theorem 18 in Chapter I], there exists $r > 1, C_2 > 1$ such that

$$
\left( \int_B w^r \, dm \right)^{1/r} \leq C_2 \int_B w \, dm, \quad \text{for all balls } B. \quad (5.39)
$$

Choose $\theta \in (0, 1)$ such that $\theta r^{-1} + 2(1 - \theta) = 1$. By Hölder inequality and (5.39)

$$
\int_B w \, dm \leq \left( \int_B w^r \, dm \right)^{\theta/r} \left( \int_B \sqrt{w} \, dm \right)^{2(1-\theta)} \leq C_2^{\theta} \left( \int_B w \, dm \right)^{\theta} \left( \int_B \sqrt{w} \, dm \right)^{2(1-\theta)}
$$

for all balls $B$. This immediately implies (b) with $C = C_2^{\theta/(1-\theta)}$. □
In the following result, we consider the case of Brownian motion in \( \mathbb{R} \); that is, \((X,d,m,\mathcal{E},\mathcal{F})\) is given by \(X = \mathbb{R}, d\) is the Euclidean distance, \(m\) is the Lebesgue measure, \(\mathcal{F} = W^{1,2}\) and \(\mathcal{E}(f,f) = \int |f'|^2 \, dm\).

**Theorem 5.17.** Let \((X,d,m,\mathcal{E},\mathcal{F})\) denote the MMD space corresponding to the Brownian motion on \(\mathbb{R}\). Then the family of Gaussian admissible measures is characterized by the following reverse Hölder inequality:

\[ G(X,d,m,\mathcal{E},\mathcal{F}) = \{ \mu : \mu \text{ is } A_\infty \text{ related to } m \}. \]

**Proof.** Set

\[ \tilde{G} = \left\{ g \, dm \mid \text{there exists } C \geq 1 \text{ such that for all } a < b, \text{ we have} \right\} \]

\[ (b - a)^{1/2} \left( \int_a^b g \, dm \right)^{1/2} \leq C \int_a^b \sqrt{g} \, dm \text{ and } g \not\equiv 0. \]

By Theorem 2.12 and Lemma 5.16, it suffices to show that \(\tilde{G} \subseteq G(X,d,m,\mathcal{E},\mathcal{F})\). Let \(g \, dm \in \tilde{G}\). Then consider the measures \(\mu_1 = \sqrt{g} \, dm\) and \(\mu = g \, dm\). For all \(a < b\), an easy calculation shows that

\[ \frac{\sqrt{g}}{\mu} \leq C \frac{\mu_1}{\mu} \leq C \frac{\mu_1}{\mu}. \]

Since \(\sqrt{g}\) satisfies a reverse Hölder inequality, by [ST, Lemma 12 and Theorem 17 in Chapter I], \(\mu_1\) is a doubling measure on \((X,d)\). By the correspondence between doubling measures and quasisymmetric maps on \(\mathbb{R}\) described in [Hei, Remark 13.20 (b)] and (5.40), we have

\[ d^\mu_{\text{int}}(a,b) = \int_a^b \sqrt{g} \, dm. \]

By the reverse Hölder inequality assumption on \(\sqrt{g}\), we have

\[ \mu_1([a,b]) \leq (b - a)^{1/2} (\mu([a,b]))^{1/2} \leq C \mu_1([a,b]). \]

Since \(\mu_1\) is a doubling measure on \((X,d)\), the above estimate shows that \(\mu\) is also a doubling measure on \((X,d)\). Since \(d^\mu_{\text{int}} \in \mathcal{J}(X,d)\), \(\mu\) is a doubling measure in \((X,d^\mu_{\text{int}})\).

By [ST, Theorem 18 of Chapter I] and Lemma 5.16, the measures \(\mu\) and \(m\) are mutually absolutely continuous. This implies that \(\mu \in \mathcal{A}(X,d,m,\mathcal{E},\mathcal{F})\).

By [CW, Theorem 1.4], for any interval \(I = [a,b]\) and for all \(f \in W^{1,2}\), we have a Poincaré inequality

\[ \int_I \left( f(x) - \frac{1}{\mu(I)} \int_I f \, d\mu \right)^2 \mu(dx) \leq K_{\mu,I}^2 \int_I |f'(x)|^2 \, dm(x), \]

where the optimal constant \(K_{\mu,I}\) satisfies the two-sided estimate

\[ K_{\mu,I} \leq \frac{1}{\mu(I)} \left( \sup_{a < x < b} \left\{ \mu([x,b])^{1/2} \left( \int_a^x \mu([a,t])^2 \, dt \right)^{1/2} \right\} \right) \]

\[ + \sup_{a < x < b} \left\{ \mu([a,x])^{1/2} \left( \int_x^b \mu([t,b])^2 \, dt \right)^{1/2} \right\} \]

\[ (5.42) \]

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By using the bound $\mu(A) \leq \mu(I)$ for all $A \subset I$, (5.42) and reverse Hölder inequality, we have

$$K_{\mu, I}^2 \lesssim \mu(I)m(I) \lesssim \mu_1(I)^2.$$  \hspace{1cm} (5.43)

By (5.40), $\mu_1(I)$ is the diameter of $I$ under the intrinsic metric $d_{\text{int}}^\mu$. Therefore by (5.42) and (5.43), we have the Poincaré inequality PI(2) for the MMD space $(X, d_{\text{int}}^\mu, \mu, E^\mu, F^\mu)$. Since $\mu$ is a doubling measure in $(X, d_{\text{int}}^\mu)$, we have that $\mu \in G(X, d, m, E, F)$.

\hspace{1cm} \Box

**Remark 5.18.** A major obstruction to determine the Gaussian admissible measures for multidimensional Brownian motion in $\mathbb{R}^n, n \geq 2$ is that the intrinsic metric with respect to $\mu = g dm$ does not admit a simple description unlike the one dimensional case where there is a simple formula (5.40). As noted in Example 5.13, the conclusion of Theorem 5.17 fails in higher dimensions.

### 6 The attainment problem for self-similar sets

In this section, we study the attainment problem, that of whether the value $d_{cw} = 2$ of the conformal walk dimension is attained, in the case of a self-similar Dirichlet form $(E, F)$ on a post-critically finite self-similar set $K$. After introducing the framework of such a Dirichlet form in Subsection 6.1, we prove in Subsection 6.2 that $d_{cw} = 2$ is attained (if and) only if $(K, d_h, \Gamma(h, h), E, F)$ satisfies PHI(2) for some harmonic function $h \in F$ and a metric $d_h$ on $K$ quasisymmetric to the resistance metric $R_E$ of $(E, F)$, where $\Gamma(h, h)$ denotes the $E$-energy measure of $h$. Then in Subsection 6.3 we present several examples, all of which are shown NOT to attain $d_{cw} = 2$, except for the two-dimensional standard Sierpiński gasket which is known to attain $d_{cw} = 2$ by Kigami’s result [Kig08, Theorem 6.3].

The restriction of the framework to post-critically finite self-similar sets is mainly for the sake of simplicity. In fact, all the results in Subsection 6.2 can be verified, with just slight modifications in the proofs, also for the canonical self-similar Dirichlet form on any generalized Sierpiński carpet, which forms essentially the only class of examples of infinitely ramified self-similar fractals where the theory of a canonical self-similar Dirichlet form has been established. We treat the case of generalized Sierpiński carpets in Subsection 6.4 and explain what changes are needed for the arguments in Subsections 6.1 and 6.2 to go through in this case.

#### 6.1 Preliminaries

In this subsection, we first introduce our framework of a post-critically finite self-similar set and a self-similar Dirichlet form on it, and then present preliminary facts.

Let us start with the standard notions concerning self-similar sets. We refer to [Kig01, Chapter 1] for details. Throughout this subsection, we fix a compact metrizable topological space $K$, a finite set $S$ with $\#S \geq 2$ and a continuous injective map $F_i : K \to K$ for each $i \in S$. We set $\mathcal{L} := (K, S, \{F_i\}_{i \in S})$. 

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Definition 6.1. (1) Let $W_0 := \{\emptyset\}$, where $\emptyset$ is an element called the empty word, let $W_n := S^n = \{w_1 \ldots w_n \mid w_i \in S \text{ for } i \in \{1, \ldots, n\}\}$ for $n \in \mathbb{N}$ and let $W_* := \bigcup_{n \in \mathbb{N} \cup \{0\}} W_n$. For $w \in W_*$, the unique $n \in \mathbb{N} \cup \{0\}$ with $w \in W_n$ is denoted by $|w|$ and called the length of $w$. For $i \in S$ and $n \in \mathbb{N} \cup \{0\}$ we write $i^n := i \ldots i \in W_n$.

(2) We set $\Sigma := S^\mathbb{N} = \{w_1 \omega_2 \omega_3 \ldots \mid \omega_i \in S \text{ for } i \in \mathbb{N}\}$, which is always equipped with the product topology of the discrete topology on $S$, and define the shift map $\sigma : \Sigma \to \Sigma$ by $\sigma(w_1 \omega_2 \omega_3 \ldots) := \omega_2 \omega_3 \omega_4 \ldots$. For $i \in S$ we define $\sigma_i : \Sigma \to \Sigma$ by $\sigma_i(w_1 \omega_2 \omega_3 \ldots) := i \omega_1 \omega_2 \omega_3 \ldots$. For $\omega = \omega_1 \omega_2 \omega_3 \ldots \in \Sigma$ and $n \in \mathbb{N} \cup \{0\}$, we write $[\omega]_n := \omega_1 \ldots \omega_n \in W_n$.

(3) For $w = w_1 \ldots w_n \in W_*$, we set $F_w := F_{w_1} \circ \cdots \circ F_{w_n}$ ($F_0 := \text{id}_K$), $K_w := F_w(K)$, $\sigma_w := \sigma_{w_1} \circ \cdots \circ \sigma_{w_n}$ ($\sigma_\emptyset := \text{id}_\Sigma$) and $\Sigma_w := \sigma_w(\Sigma)$, and if $w \neq \emptyset$ then $w^\infty \in \Sigma$ is defined by $w^\infty := \omega w \omega \ldots$ in the natural manner.

Definition 6.2. $\mathcal{L} = (K, \Sigma, \{F_i\}_{i \in S})$ is called a self-similar structure if and only if there exists a continuous surjective map $\pi : \Sigma \to K$ such that $F_i \circ \pi = \pi \circ \sigma_i$ for any $i \in S$. Note that such $\pi$, if it exists, is unique and satisfies $\{\pi(\omega)\} = \bigcap_{n \in \mathbb{N}} K_{[\omega]_n}$ for any $\omega \in \Sigma$.

In the rest of this subsection we always assume that $\mathcal{L}$ is a self-similar structure.

Definition 6.3. (1) We define the critical set $\mathcal{C}_\mathcal{L}$ and the post-critical set $\mathcal{P}_\mathcal{L}$ of $\mathcal{L}$ by

$$
\mathcal{C}_\mathcal{L} := \pi^{-1}\left(\bigcup_{i,j \in S, i \neq j} K_i \cap K_j\right) \quad \text{and} \quad \mathcal{P}_\mathcal{L} := \bigcup_{n \in \mathbb{N}} \sigma^n(\mathcal{C}_\mathcal{L}).
$$

$\mathcal{L}$ is called post-critically finite, or p.-c.f. for short, if and only if $\mathcal{P}_\mathcal{L}$ is a finite set.

(2) We set $V_0 := \pi(\mathcal{P}_\mathcal{L})$, $V_n := \bigcup_{w \in W_n} F_w(V_0)$ for $n \in \mathbb{N}$ and $V_* := \bigcup_{n \in \mathbb{N} \cup \{0\}} V_n$.

$V_0$ should be considered as the “boundary” of the self-similar set $K$; indeed, $K_w \cap K_v = F_w(V_0) \cap F_v(V_0)$ for any $w, v \in W_*$ with $\Sigma_w \cap \Sigma_v = \emptyset$ by [Kig01, Proposition 1.3.5-(2)]. According to [Kig01, Lemma 1.3.11], $V_{n-1} \subset V_n$ for any $n \in \mathbb{N}$, and if $V_0 \neq \emptyset$ then $V_*$ is dense in $K$. Also note that by [Kig01, Theorem 1.6.2], $K$ is connected if and only if any $i, j \in S$ admit $n \in \mathbb{N}$ and $\{i_k\}_{k=0}^n \subset S$ with $i_0 = i$ and $i_n = j$ such that $K_{i_{k-1}} \cap K_{i_k} \neq \emptyset$ for any $k \in \{1, \ldots, n\}$, and if $K$ is connected then it is arcwise connected.

In the remainder of this subsection our self-similar structure $\mathcal{L} = (K, \Sigma, \{F_i\}_{i \in S})$ is always assumed to be post-critically finite with $K$ connected, so that $2 \leq |V_0| < \infty$, $K \equiv V_0 = V_0$ and $V_*$ is countably infinite and dense in $K$.

Next we briefly recall the construction and basic properties of a self-similar Dirichlet form on such $\mathcal{L}$; see [Kig01, Chapter 3] for details. Let $D = (D_{pq})_{p,q \in V_0}$ be a real symmetric matrix of size $|V_0|$ (which we also regard as a linear operator on $\mathbb{R}^{V_0}$) such that

(D1) $\{u \in \mathbb{R}^{V_0} \mid Du = 0\} = \mathbb{R}1_{V_0}$,
(D2) $D_{pq} \geq 0$ for any $p, q \in V_0$ with $p \neq q$.

We define $\mathcal{E}^{(0)}(u, v) := -\sum_{p,q \in V_0} D_{pq} u(q) v(p)$ for $u, v \in \mathbb{R}^{V_0}$, so that $(\mathcal{E}^{(0)}, \mathbb{R}^{V_0})$ is a Dirichlet form on $L^2(V_0, \#)$. Furthermore let $r = (r_i)_{i \in S} \in (0, \infty)^S$ and define

$$
\mathcal{E}^{(n)}(u, v) := \sum_{w \in W_n} \frac{1}{r_w} \mathcal{E}^{(0)}(u \circ F_w|_{V_0}, v \circ F_w|_{V_0}), \quad u, v \in \mathbb{R}^{V_0}.
$$

(6.2)
for each \( n \in \mathbb{N} \), where \( r_w := r_{w_1} r_{w_2} \ldots r_{w_n} \) for \( w = w_1 w_2 \ldots w_n \in W_n \) (\( r_0 := 1 \)).

**Definition 6.4.** The pair \((D, r)\) of a real symmetric matrix \( D = (D_{pq})_{p,q \in V_0} \) of size \( \#V_0 \) with the properties (D1) and (D2) and \( r = (r_i)_{i \in S} \in (0, \infty)^S \) is called a *harmonic structure* on \( L \) if and only if \( E^{(0)}(u, u) = \inf_{v \in \mathbb{R}^{V_1}, v|_{V_0} = u} E^{(1)}(v, v) \) for any \( u \in \mathbb{R}^{V_0} \); note that then

\[
E^{(n)}(u, u) = \min_{v \in \mathbb{R}^{V_{1+n}}, v|_{V_0} = u} E^{(n+1)}(v, v)
\]

for any \( n_1, n_2 \in \mathbb{N} \cup \{0\} \) with \( n_1 \leq n_2 \) and any \( u \in \mathbb{R}^{V_{n_1}} \) by [Kig01, Proposition 3.1.3]. If \( r \in (0, 1)^S \) in addition, then \((D, r)\) is called *regular*.

In the rest of this subsection, we assume that \((D, r)\) is a regular harmonic structure on \( L \). In this case, \( \{E^{(n)}(u|_{V_n}, u|_{V_n})\}_{n \in \mathbb{N} \cup \{0\}} \) is non-decreasing and hence has the limit in \([0, \infty]\) for any \( u \in C(K) \). Then we define a linear subspace \( F \) of \( C(K) \) and a non-negative definite symmetric bilinear form \( E : F \times F \to \mathbb{R} \) by

\[
F := \{ u \in C(K) \mid \lim_{n \to \infty} E^{(n)}(u|_{V_n}, u|_{V_n}) < \infty \},
\]

\[
E(u, v) := \lim_{n \to \infty} E^{(n)}(u|_{V_n}, v|_{V_n}) \in \mathbb{R}, \quad u, v \in F,
\]

so that \((E, F)\) is easily seen to possess the following self-similarity properties (note that \( F \cap C(K) = F \) in the present setting):

\[
F \cap C(K) = \{ u \in C(K) \mid u \circ F_i \in F \text{ for any } i \in S \},
\]

\[
E(u, v) = \sum_{i \in S} \frac{1}{r_i} E(u \circ F_i, v \circ F_i), \quad u, v \in F \cap C(K).
\]

By [Kig01, Proposition 2.2.4, Lemma 2.2.5, Theorem 2.2.6, Lemma 2.3.9, Theorems 2.3.10 and 3.3.4], \((E, F)\) is a resistance form on \( K \) and its resistance metric \( R := K \times K \to [0, \infty) \) is a metric on \( K \) compatible with the original topology of \( K \); here the former means that \((E, F)\) has the following properties (see [Kig01, Definition 2.3.1] or [Kig12, Definition 3.1]):

(RF1) \( \{ u \in F \mid E(u, u) = 0 \} = \mathbb{R} 1_K \).

(RF2) \((F/\mathbb{R} 1_K, E)\) is a Hilbert space.

(RF3) \( \{ u|_V \mid u \in F \} = \mathbb{R}^V \) for any non-empty finite subset \( V \) of \( K \).

(RF4) \( R(x, y) := \sup_{u \in F \setminus \mathbb{R} 1} |u(x) - u(y)|^2/E(u, u) < \infty \) for any \( x, y \in K \).

(RF5) \( u^+ \land 1 \in F \) and \( E(u^+ \land 1, u^+ \land 1) \leq E(u, u) \) for any \( u \in F \).

See [Kig01, Chapter 2] and [Kig12, Part I] for further details of resistance forms.

In the present framework, the notion of harmonic functions is defined as follows.

**Definition 6.5.** Let \( n \in \mathbb{N} \cup \{0\} \). A continuous function \( h \in C(K) \) is called *\( E \)-harmonic* on \( K \setminus V_n \), or *\( n \)-harmonic* for short, if and only if \( h \in F \) and

\[
E(h, h) = \inf_{v \in F, v|_{V_n} = h|_{V_n}} E(v, v), \quad \text{or equivalently, } \ E(h, v) = 0 \text{ for any } v \in F^{K \setminus V_n},
\]

where \( F^{K \setminus V_n} := \{ u \in F \mid u|_{V_n} = 0 \} \). We set \( \mathcal{H}_n := \{ h \in C(K) \mid h \text{ is } n \text{-harmonic} \} \).
It is obvious that \( \mathcal{H}_n \) is a linear subspace of \( \mathcal{F} \) and \( \mathbb{R}^1_K \subset \mathcal{H}_n \subset \mathcal{H}_{n+1} \) for any \( n \in \mathbb{N} \cup \{0\} \). Moreover, we easily have the following proposition by [Kig01, Lemma 2.2.2 and Theorem 3.2.4], (6.2), (6.3), (6.6) and (6.7).

**Proposition 6.6.** Let \( n \in \mathbb{N} \cup \{0\} \).

1. For each \( u \in \mathbb{R}^V \) there exists a unique \( H_n(u) \in \mathcal{H}_n \) such that \( H_n(u)|_{V_n} = u \). Moreover, \( H_n : \mathbb{R}^V \to \mathcal{H}_n \) is linear (and hence it is a linear isomorphism).
2. It holds that
   \[
   \mathcal{H}_n = \{ h \in \mathcal{F} \mid \mathcal{E}(h, h) = \mathcal{E}^{(n)}(h|_{V_n}, h|_{V_n}) \} = \{ h \in C(K) \mid h \circ F_w \in \mathcal{H}_0 \text{ for any } w \in W_n \}. 
   \]
   In particular, for each \( w \in W_n \), a linear map \( F_w^* : \mathcal{H}_0 \to \mathcal{H}_0 \) is defined by \( F_w^* h := h \circ F_w \).

Now we equip \( K \) with a measure to turn \((\mathcal{E}, \mathcal{F})\) into a Dirichlet form. Indeed, we have the following proposition.

**Proposition 6.7.** Let \( \mu \) be a Radon measure on \( K \) with full support. Then \((\mathcal{E}, \mathcal{F})\) is an irreducible, strongly local, regular symmetric Dirichlet form on \( L^2(K, \mu) \), and its extended Dirichlet space \( \mathcal{F}_e \) coincides with \( \mathcal{F} \). Moreover, the capacity \( \text{Cap}^\mu \) associated with \( (K, R_\mathcal{E}, \mu, \mathcal{E}, \mathcal{F}) \) satisfies \( \inf_{x \in K} \text{Cap}^\mu(\{x\}) > 0 \), and in particular (recall Definition 2.9)
   \[
   \mathcal{A}(K, R_\mathcal{E}, \mu, \mathcal{E}, \mathcal{F}) = \{ \nu \mid \nu \text{ is a Radon measure on } K \text{ with full support} \}. 
   \]

**Proof.** \((\mathcal{E}, \mathcal{F})\) is a regular symmetric Dirichlet form on \( L^2(K, \mu) \) by [Kig12, Corollary 6.4 and Theorem 9.4], strongly local by the same argument as [Hin05, Proof of Lemma 3.12] on the basis of (6.6), (6.7) and \( \mathcal{E}(1_K, 1_K) = 0 \), and irreducible by (RF1) above and [CF, Theorem 2.1.11]. The equality \( \mathcal{F}_e = \mathcal{F} \) is immediate from (RF1), (RF2) and (RF4). We also easily see from (RF4), \( \text{diam}(K, R_\mathcal{E}) < \infty \) and \( \mu(K) < \infty \) that \( \inf_{x \in K} \text{Cap}^\mu(\{x\}) > 0 \), so that a subset of \( K \) is quasi-closed with respect to \( (K, R_\mathcal{E}, \mu, \mathcal{E}, \mathcal{F}) \) if and only if it is closed in \( K \). In particular, any Radon measure \( \nu \) on \( K \) is smooth with respect to \( (K, R_\mathcal{E}, \mu, \mathcal{E}, \mathcal{F}) \) and the only quasi-support of \( \nu \) with respect to \( (K, R_\mathcal{E}, \mu, \mathcal{E}, \mathcal{F}) \) is the support of \( \nu \) in \( K \), which together imply (6.11). \( \square \)

Let \( d_H \in (0, \infty) \) be such that \( \sum_{i \in S} r^d_i = 1 \), so that \( d_H \geq 1 \) since
   \[
   \max_{x,y \in V_0} R_\mathcal{E}(x, y) \leq \sum_{i \in S} \max_{x,y \in V_0} R_\mathcal{E}(F_i(x), F_i(y)) \leq \left( \sum_{i \in S} r_i \right) \max_{x,y \in V_0} R_\mathcal{E}(x, y) 
   \]
by the connectedness of \( K \), [Kig01, Theorem 1.6.2 and Lemma 3.3.5] and hence \( \sum_{i \in S} r_i \geq 1 \). Let \( m \) be the self-similar measure on \( \mathcal{L} \) with weight \( (r^d_i)_{i \in S} \), i.e., the unique Borel measure on \( K \) such that \( m(K_w) = r^d_i \) for any \( w \in W_n \). The measure \( m \) could be considered as the “uniform distribution” on \( \mathcal{L} \), and it is the most typical choice of the reference measure \( \mu \) for \((\mathcal{E}, \mathcal{F})\). It is well known that \((K, R_\mathcal{E}, m, \mathcal{E}, \mathcal{F})\) satisfies \( \text{PHI}(d_H+1) \); more precisely, the following lemma and proposition hold.

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Lemma 6.8. There exist $c_1, c_2 \in (0, \infty)$ such that for any $(x, s) \in K \times (0, \text{diam}(K, R_\varepsilon)]$, \[c_1 s^{d_H} \leq m(B_{R_\varepsilon}(x, s)) \leq c_2 s^{d_H}. \quad (6.13)\]

Proof. This is immediate from Lemma 6.14 below and [Kig01, Lemma 4.2.3]. \qed

Proposition 6.9. $(K, R_\varepsilon, m, \varepsilon, F)$ satisfies PHI($d_H + 1$).

Proof. Lemma 6.8 and [Kig12, Theorem 15.10] together imply that $(K, R_\varepsilon, m, \varepsilon, F)$ satisfies HKE($d_H + 1$) as well as VD and RVD, and therefore it also satisfies PHI($d_H + 1$) by Theorem 4.4. \qed

We conclude this subsection with the following Proposition, which is essentially due to Kigami [Kig09, Kig12] and gives a simple equivalent condition for the validity of PHI($\beta$) after quasisymmetric change of the metric and time change.

Proposition 6.10. Let $\theta$ be a metric on $K$ quasisymmetric to $R_\varepsilon$, let $\mu$ be a Radon measure on $K$ with full support and let $\beta \in [2, \infty)$. Then the following conditions are equivalent:

(a) $(K, \theta, \mu, \varepsilon, F)$ satisfies PHI($\beta$).

(b) There exists $C \in (1, \infty)$ such that \[C^{-1}(\text{diam}(K_w, \theta))^{\beta} \leq r_w \mu(K_w) \leq C(\text{diam}(K_w, \theta))^{\beta} \quad \text{for any } w \in W_* . \quad (6.14)\]

Moreover, if either of these conditions holds, then $\mu(F_w(V_0)) = 0$ for any $w \in W_*$ and $\mu(\{x\}) = 0$ for any $x \in K$.

The rest of this subsection is devoted to the proof of Proposition 6.10, which requires the following lemmas and definitions.

Lemma 6.11 ([Kig01, Lemma 3.3.5], [Kig04, Theorem A.1]). There exists $c_{R_\varepsilon} \in (0, 1]$ such that for any $w \in W_*$ and any $x, y \in K$, \[c_{R_\varepsilon} r_w R_\varepsilon(x, y) \leq R_\varepsilon(F_w(x), F_w(y)) \leq r_w R_\varepsilon(x, y). \quad (6.15)\]

Definition 6.12. (1) Let $w, v \in W_*$, $w = w_1 \ldots w_{n_1}, v = v_1 \ldots v_{n_2}$. We define $wv \in W_*$ by $wv := w_1 \ldots w_{n_1} v_1 \ldots v_{n_2}$ ($w\emptyset := w, \emptyset v := v$). We write $w \leq v$ if and only if $w = v\tau$ for some $\tau \in W_*$; note that $\Sigma_w \cap \Sigma_v = \emptyset$ if and only if neither $w \leq v$ nor $v \leq w$.

(2) A finite subset $\Lambda$ of $W_*$ is called a partition of $\Sigma$ if and only if $\Sigma_w \cap \Sigma_v = \emptyset$ for any $w, v \in \Lambda$ with $w \neq v$ and $\Sigma = \bigcup_{w \in \Lambda} \Sigma_w$.

(3) Let $\Lambda_1, \Lambda_2$ be partitions of $\Sigma$. We say that $\Lambda_1$ is a refinement of $\Lambda_2$, and write $\Lambda_1 \leq \Lambda_2$, if and only if for each $w^1 \in \Lambda_1$ there exists $w^2 \in \Lambda_2$ such that $w^1 \leq w^2$. 

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Definition 6.13. (1) We define $\Lambda_1 := \{\emptyset\}$,

$$\Lambda_s := \{w \mid w = w_1 \ldots w_n \in W_s \setminus \{\emptyset\}, r_{w_1 \ldots w_{n-1}} > s \geq r_w\} \quad (6.16)$$

for each $s \in (0,1)$, and $S := \{\Lambda_s\}_{s \in [0,1]}$. We call $S$ the scale on $\Sigma$ associated with $(\mathcal{L}, D, r)$.

(2) For each $(s, x) \in (0,1] \times K$, we define $\Lambda_{s,x} := \{w \in \Lambda_s \mid x \in K_w\}$, $K_s(x) := \bigcup_{w \in \Lambda_{s,x}} K_w$, $\Lambda_s^1 := \{w \in \Lambda_s \mid K_w \cap K_s(x) \neq \emptyset\}$ and $U_s(x) := \bigcup_{w \in \Lambda_{s,x}^1} K_w$.

Clearly $\lim_{s \downarrow 0} \min\{|w| \mid w \in \Lambda_s\} = \infty$, and it is easy to see that $\Lambda_s$ is a partition of $\Sigma$ for any $s \in (0,1]$ and that $\Lambda_{s_1} \leq \Lambda_{s_2}$ for any $s_1, s_2 \in (0,1]$ with $s_1 \leq s_2$. These facts together with [Kig01, Proposition 1.3.6] imply that for any $x \in K$, each of $\{K_s(x)\}_{s \in (0,1]}$ and $\{U_s(x)\}_{s \in (0,1]}$ is non-decreasing in $s$ and forms a fundamental system of neighborhoods of $x$ in $K$. Moreover, $(U_s(x))_{(s,x) \in (0,1] \times K}$ can be used as a replacement for the metric balls $(B_{R_\varepsilon}(x,s))_{(s,x) \in K \times [0,\text{diam}(K,R_\varepsilon)]}$ in $(K,R_\varepsilon)$ by virtue of the following lemma.

Lemma 6.14. There exist $\alpha_1, \alpha_2 \in (0,\infty)$ such that for any $(s, x) \in (0,1] \times K$,

$$B_{R_\varepsilon}(x,\alpha_1 s) \subset U_s(x) \subset B_{R_\varepsilon}(x,\alpha_2 s). \quad (6.17)$$

Proof. By the upper inequality in (6.15) we have $\text{diam}(K_w, R_\varepsilon) \leq r_w \text{diam}(K, R_\varepsilon)$ for any $w \in W_s$, which implies the latter inclusion in (6.17) with $\alpha_2 \in (2\text{diam}(K, R_\varepsilon), \infty)$ arbitrary. On the other hand, by [Kig01, Proof of Lemma 4.2.4] there exists $\alpha_1 \in (0,\infty)$ such that $R_\varepsilon(x,y) \geq \alpha_1 s$ for any $s \in (0,1]$, any $w, v \in \Lambda_s$ with $K_w \cap K_v = \emptyset$ and any $(x,y) \in K_w \times K_v$, which yields the former inclusion in (6.17).

Proof. [Proof of Proposition 6.10] This equivalence can be easily concluded by combining Theorem 4.4 and results in [Kig12, Kig09], as follows. First, by Theorem 4.4 and [Kig12, Theorem 15.10], under the quasisymmetry of $\theta$ to $R_\varepsilon$, (a) is equivalent to the following condition (c):

(c) $(K, \theta, \mu)$ is VD and there exists $C \in (0,\infty)$ such that for any $x, y \in K$ with $x \neq y$,

$$C^{-1} \theta(x,y)^{\beta} \leq R_\varepsilon(x,y) \mu(B_{\theta}(x,\theta(x,y))) \leq C \theta(x,y)^{\beta}. \quad (6.18)$$

Next, by the quasisymmetry of $\theta$ to $R_\varepsilon$ again, (3.1) and (3.3), $(K, \theta, \mu)$ is VD if and only if $(K, R_\varepsilon, \mu)$ is VD, which in turn is, by Lemma 6.14 and the compactness of $K$, equivalent to the existence of $C \in (0,\infty)$ such that

$$\mu(U_s(x)) \leq C \mu(U_{s/2}(x)) \quad \text{for any } (s,x) \in (0,1] \times K. \quad (6.19)$$

Then by [Kig09, Theorem 1.3.5] and the fact that $S$ is locally finite with respect to $\mathcal{L}$, i.e.,

$$\sup_{(s,x) \in (0,1] \times K} \# \Lambda_{s,x}^1 < \infty \quad (6.20)$$
by [Kig01, Lemma 4.2.3] and [Kig09, Lemma 1.3.6], we have (6.19) if and only if there exists $C \in (1, \infty)$ such that the following hold:

$$\mu(K_{wj}) \geq C^{-1} \mu(K_w) \quad \text{for any } (w, j) \in W_s \times S,$$

$$\mu(K_w) \leq C \mu(K_v) \quad \text{for any } s \in (0,1] \text{ and any } w, v \in \Lambda_s \text{ with } K_w \cap K_v \neq \emptyset.$$  

Moreover, by [Kig09, Theorem 1.2.4], (6.21) implies that

$$\mu(F_w(V_0)) = 0 = \mu(\{x\}) \quad \text{for any } w \in W_s \text{ and any } x \in K.$$  

(We remark that (6.23) is part of the assumptions of [Kig09, Theorem 1.3.5] but can be dropped; indeed, even without assuming (6.23), [Kig09, Proofs of Theorems 1.3.10 and 1.3.11] show that any one of the three conditions [Kig09, Theorem 1.3.5-(1),(2),(3)] implies (6.21), from which (6.23) also follows by [Kig09, Theorem 1.2.4].)

On the other hand, since the quasisymmetry of $\theta$ to $R_{\varepsilon}$ yields $\delta_1, \delta_2 \in (0, \infty)$ such that $B_0(x, \delta_1\theta(x, y)) \subset B_{R_{\varepsilon}}(x, R_{\varepsilon}(x, y)) \subset B_0(x, \delta_2\theta(x, y))$ for any $x, y \in K$ with $x \neq y$ by (3.1), under VD of $(K, \theta, \mu)$ and $(K, R_{\varepsilon}, \mu)$ we have (6.18) if and only if there exists $C \in (0, \infty)$ such that for any $x, y \in K$ with $x \neq y$,

$$C^{-1}\theta(x, y)^{\beta} \leq R_{\varepsilon}(x, y) \mu(B_{R_{\varepsilon}}(x, R_{\varepsilon}(x, y))) \leq C \theta(x, y)^{\beta}. \quad (6.24)$$

Therefore (c) is equivalent to the following condition (d):

(d) There exists $C \in (0, \infty)$ such that (6.21), (6.22) and (6.24) hold.

Thus it remains to show that (d) is equivalent to (b). Indeed, let $w \in W_s$, take $x, y \in K_w$ with the property $\text{diam}(K_w, R_{\varepsilon}) = R_{\varepsilon}(x, y)$, and note that $R_{\varepsilon}$ is quasisymmetric to $\theta$ by [Hei, Proposition 10.6], so that $w \in \Lambda_{r_w}, R_{\varepsilon}(x, y)/\text{diam}(K, R_{\varepsilon}) \in [c_{R_{\varepsilon}} r_w, r_w]$ by Lemma 6.11, and $\theta(x, y)^{\beta} \asymp (\text{diam}(K_w, \theta))^{\beta}$ by (3.5). If (d) holds, then since $(K, R_{\varepsilon}, \mu)$ is VD we easily see from Lemma 6.14, (6.20) and (6.22) that

$$\mu(B_{R_{\varepsilon}}(x, \text{diam}(K_w, R_{\varepsilon}))) = \mu(B_{R_{\varepsilon}}(x, R_{\varepsilon}(x, y))) \asymp \mu(K_w) \quad (6.25)$$

and hence (6.24) implies (b). Conversely suppose that (b) holds. Then for any $j \in S, Lemma 6.11$ yields $\text{diam}(K_{wj}, R_{\varepsilon})/\text{diam}(K_w, R_{\varepsilon}) \in [c_{R_{\varepsilon}} \min_{k \in S} r_k, c_{R_{\varepsilon}}^{-1} \max_{k \in S} r_k]$, hence

$$(\text{diam}(K_{wj}, \theta))^{\beta} \asymp (\text{diam}(K_w, \theta))^{\beta} \quad (6.26)$$

by the quasisymmetry of $R_{\varepsilon}$ to $\theta$ and (3.5), and therefore (6.14) implies $\mu(K_{wj}) \asymp \mu(K_w)$, i.e., (6.21) holds. Also for any $s \in (0,1]$ and any $v, \tau \in \Lambda_s$ with $K_v \cap K_\tau \neq \emptyset$, $\text{diam}(K_v, R_{\varepsilon}) \asymp \text{diam}(K_v \cup K_\tau, R_{\varepsilon}) \asymp \text{diam}(K_\tau, R_{\varepsilon})$ by Lemma 6.11 and hence

$$(\text{diam}(K_v, \theta))^{\beta} \asymp (\text{diam}(K_v \cup K_\tau, \theta))^{\beta} \asymp (\text{diam}(K_\tau, \theta))^{\beta} \quad (6.27)$$

by the quasisymmetry of $R_{\varepsilon}$ to $\theta$ and (3.5), which together with (6.14) implies that $\mu(K_v) \asymp \mu(K_\tau)$, proving (6.22). In particular, $(K, R_{\varepsilon}, \mu)$ is VD, and now it follows from Lemma 6.14, (6.20) and (6.22) that (6.25) holds, which together with (6.14) yields (6.24), proving (d).
6.2 A necessary condition: attainment by the energy measure of some harmonic function

Throughout this subsection, we assume that \( \mathcal{L} = (K, S, \{F_i\}_{i \in S}) \) is a post-critically finite self-similar structure with \( \#S \geq 2 \) and \( K \) connected and that \((D, r)\) is a regular harmonic structure on \( \mathcal{L} \), and we follow the notation introduced in Subsection 6.1.

Proposition 6.10 with \( \beta = 2 \) justifies introducing the following set of pairs of metrics and Borel probability measures on \( K \).

**Definition 6.15.** We set \( \text{Homeo}^+ := \{\eta \mid \eta : [0, \infty) \to [0, \infty), \eta \text{ is a homeomorphism}\} \), set \( \mathcal{P}(K) := \{\mu \mid \mu \text{ is a Borel probability measure on } K\} \), which is equipped with the topology of weak convergence, and for each \( \eta \in \text{Homeo}^+ \) and \( C \in (1, \infty) \) we define

\[
\mathcal{M}_2(\eta, C) := \mathcal{M}_2(\mathcal{L}, D, r;\eta, C)
\]

\[
:= \left\{ (\theta, \mu) \mid \theta \text{ is a metric on } K \text{ which is } \eta\text{-quasisymmetric to } R_\varepsilon, \mu \in \mathcal{P}(K), C^{-1}(\text{diam}(K_w, \theta))^2 \leq r_w \mu(K_w) \leq C(\text{diam}(K_w, \theta))^2 \text{ for any } w \in W^* \right\},
\]

(6.28)

which is considered as a subset of \( C(K \times K) \times \mathcal{P}(K) \). We also set \( \mathcal{M}_2 := \bigcup_{\eta \in \text{Homeo}^+, C \in (1, \infty)} \mathcal{M}_2(\eta, C) \).

Since \( \mu \in A(K, R_\varepsilon, m, \mathcal{E}, \mathcal{F}) \) for any \((\theta, \mu) \in \mathcal{M}_2 \) by (6.11) with \( m \) in place of \( \mu \) and (6.28), Proposition 6.10 with \( \beta = 2 \) means that

for the MMD space \((K, R_\varepsilon, m, \mathcal{E}, \mathcal{F})\) the value \( d_{cw} = 2 \) is attained by \((K, \theta, \mu, \mathcal{E}, \mathcal{F})\) for some \( \theta \in \mathcal{J}(K, R_\varepsilon) \) and some \( \mu \in A(K, R_\varepsilon, m, \mathcal{E}, \mathcal{F}) \) if and only if \( \mathcal{M}_2 \neq \emptyset \).

In fact, it turns out that in this case the value \( d_{cw} = 2 \) has to be attained by \((\theta_h, \Gamma(h, h))\) for some \( h \in H_0 \setminus \mathbb{R}1_K \) and some \( \theta_h \in \mathcal{J}(K, R_\varepsilon) \), which is the main result of this subsection and stated as follows. We take arbitrary \( \eta \in \text{Homeo}^+ \) and \( C \in (1, \infty) \), define \( \tilde{\eta} \in \text{Homeo}^+ \) by \( \tilde{\eta}(t) := 1/\eta^{-1}(t^{-1}) \) \((\tilde{\eta}(0) := 0)\) and fix them throughout the rest of this subsection.

**Theorem 6.16.** If \( \mathcal{M}_2(\eta, C) \neq \emptyset \), then there exist \( h \in H_0 \setminus \mathbb{R}1_K \) and a metric \( \theta_h \) on \( K \) such that \((\theta_h, \Gamma(h, h)) \in \mathcal{M}_2(c_{R_\varepsilon}^{-1} \eta, C)\), where \( c_{R_\varepsilon} \) is the constant in Lemma 6.11.

The rest of this subsection is devoted to the proof of Theorem 6.16, which is reduced to proving a series of propositions and lemmas concerning the set \( \mathcal{M}_2(\eta, C) \). We start with establishing its compactness. Note that \( \mathcal{P}(K) \) is a compact metrizable topological space by [Str, Theorems 9.1.5 and 9.1.9] and hence that \( C(K \times K) \times \mathcal{P}(K) \) is also metrizable.

**Proposition 6.17.** \( \mathcal{M}_2(\eta, C) \) is a compact subset of \( C(K \times K) \times \mathcal{P}(K) \).

**Proof.** Let \( \{(\theta_n, \mu_n)\}_{n \in \mathbb{N}} \subset \mathcal{M}_2(\eta, C) \). By the metrizability of \( C(K \times K) \times \mathcal{P}(K) \) noted above, it suffices to show that there exists a subsequence of \( \{(\theta_n, \mu_n)\}_{n \in \mathbb{N}} \) converging to some \((\theta, \mu) \in \mathcal{M}_2(\eta, C) \) in \( C(K \times K) \times \mathcal{P}(K) \).
First, recalling that the compactness of $K$ implies that of $\mathcal{P}(K)$ by [Str, Theorem 9.1.9], we can choose $\mu \in \mathcal{P}(K)$ and a subsequence of $\{\mu_n\}_{n \in \mathbb{N}}$ converging to $\mu$ in $\mathcal{P}(K)$ and therefore we may assume that $\{\mu_n\}_{n \in \mathbb{N}}$ itself converges to $\mu$ in $\mathcal{P}(K)$.

Next, $\text{diam}(K, \theta_n) \in [C^{-1/2}, C^{1/2}]$ for any $n \in \mathbb{N}$ by the inequalities in (6.28) and hence $\{\theta_n\}_{n \in \mathbb{N}}$ is uniformly bounded. Moreover, for each $n \in \mathbb{N}$, since $R_\varepsilon$ is $\tilde{\eta}$-quasisymmetric to $\theta_n$ by [Hei, Proposition 10.6], it follows from (3.5) that for any $x, y \in K$,

\[
\frac{R_\varepsilon(x, y)}{\text{diam}(K, R_\varepsilon)} = \frac{\text{diam}\{x, y\}, R_\varepsilon}{\text{diam}(K, R_\varepsilon)} \leq \eta \left(2 \text{diam}\{x, y\}, \theta_n\right) \leq \eta(2C^{1/2}\theta_n(x, y)), \quad (6.30)
\]

\[
\frac{C^{-1/2}\theta_n(x, y)}{\text{diam}(K, \theta_n)} \leq \frac{\text{diam}\{x, y\}, \theta_n}{\text{diam}(K, \theta_n)} \leq \tilde{\eta} \left(2 \text{diam}\{x, y\}, R_\varepsilon\right) = \tilde{\eta} \left(\frac{2R_\varepsilon(x, y)}{\text{diam}(K, R_\varepsilon)}\right), \quad (6.31)
\]

which in turn implies that for any $x_1, y_1, x_2, y_2 \in K$,

\[
|\theta_n(x_1, y_1) - \theta_n(x_2, y_2)| \leq \theta_n(x_1, x_2) + \theta_n(y_1, y_2) \leq C^{1/2} \left(\tilde{\eta} \left(\frac{2R_\varepsilon(x_1, x_2)}{\text{diam}(K, R_\varepsilon)}\right) + \tilde{\eta} \left(\frac{2R_\varepsilon(y_1, y_2)}{\text{diam}(K, R_\varepsilon)}\right)\right), \quad (6.32)
\]

so that $\{\theta_n\}_{n \in \mathbb{N}} \subset C(K \times K)$ is equicontinuous. Thus by the Arzelà-Ascoli theorem (see, e.g., [Rud, Theorem 11.28]) there exist $\theta \in C(K \times K)$ and a subsequence $\{\theta_{n_k}\}_{k \in \mathbb{N}}$ of $\{\theta_n\}_{n \in \mathbb{N}}$ converging to $\theta$ in $C(K \times K)$, so that $\{\theta_{n_k}, \mu_{n_k}\}_{k \in \mathbb{N}}$ converges to $(\theta, \mu)$ in $C(K \times K) \times \mathcal{P}(K)$. Then $\text{diam}(K, \theta) \in [C^{-1/2}, C^{1/2}]$ and for any $x, y, z \in K$ we have (6.30) with $\theta$ in place of $\theta_n$, $\theta(x, x) = 0$, $\theta(x, y) = \theta(y, x) \geq 0$ and $\theta(x, y) \leq \theta(x, z) + \theta(z, y)$ by the same properties of $\theta_{n_k}$ for $k \in \mathbb{N}$, whence $\theta$ is a metric on $K$. Furthermore letting $k \to \infty$ in the $\tilde{\eta}$-quasisymmetry of $R_\varepsilon$ to $\theta_{n_k}$ yields that of $R_\varepsilon$ to $\theta$, which implies the $\eta$-quasisymmetry of $\theta$ to $R_\varepsilon$ by [Hei, Proposition 10.6] and the fact that $\eta(t) = 1/\tilde{\eta}^{-1}(t^{-1})$ for any $t \in (0, \infty)$.

To show the inequalities in (6.28) for $(\theta, \mu)$, let $w \in W_s$, choose $x = x_w \in K_w \setminus F_w(V_0)$ and set $s := r_w$, so that $w \in \Lambda_s$ and $\Lambda_{s, x} = \{w\}$ by [Kig01, Proposition 1.3.5-(2)] (recall Definition 6.13). Note that $K_w = F_w(K)$ is compact and hence closed in $K$, that $K_w \setminus F_w(V_0) = K \setminus (F_w(V_0) \cup \bigcup_{v \in W_s \setminus \{w\}} K_v)$ by [Kig01, Proposition 1.3.5-(2)] and is thus open in $K$, and that $K_w \subset U_s^\circ(x)$ with $U_s^\circ(x)$ the interior of $U_s(x)$ in $K$ by [Kig01, Proposition 1.3.6]. By using these facts and the convergence of $\{\theta_{n_k}, \mu_{n_k}\}_{k \in \mathbb{N}}$ to $(\theta, \mu)$ in $C(K \times K) \times \mathcal{P}(K)$, let $k \to \infty$ in the inequalities in (6.28) for $(\theta_{n_k}, \mu_{n_k})$, we obtain

\[
r_w \mu(K_w) \geq \limsup_{k \to \infty} r_w \mu_{n_k}(K_w) \geq C^{-1}(\text{diam}(K_w, \theta))^2, \quad (6.33)
\]

\[
r_w \mu(K_w \setminus F_w(V_0)) \leq \liminf_{k \to \infty} r_w \mu_{n_k}(K_w \setminus F_w(V_0)) \leq C(\text{diam}(K_w, \theta))^2, \quad (6.34)
\]

\[
r_w \mu(K_w) \leq r_w \mu(U_s^\circ(x)) \leq \liminf_{k \to \infty} r_w \mu_{n_k}(U_s^\circ(x)) \leq \sum_{v \in \Lambda_s^1} Cr_v r_v^{-1}(\text{diam}(K_v, \theta))^2
\]

\[
\leq C(\min_{k \in S} r_k)^{-1} \sum_{v \in \Lambda_s^1} (\text{diam}(K_v, \theta))^2 \approx (\text{diam}(K_w, \theta))^2, \quad (6.35)
\]

where the last step in (6.35) follows from (6.27) and (6.20). We now conclude from (6.33), (6.35) and (6.26) that $\mu(K_{w_j}) \approx r_w^{-1}(\text{diam}(K_{w_j}, \theta))^2 \approx r_w^{-1}(\text{diam}(K_w, \theta))^2 \approx \mu(K_w)$ for
any \((w, j) \in W_\ast \times S\), which together with [Kig09, Theorem 1.2.4] implies that \(\mu(F_w(V_0)) = 0\) and hence \(\mu(K_w \setminus F_w(V_0)) = \mu(K_w)\) for any \(w \in W_\ast\), so that (6.33) and (6.34) yield the inequalities in (6.28) for \((\theta, \mu)\) and thus \((\theta, \mu) \in M_2(\eta, C)\). \(\square\)

**Corollary 6.18.** Let \(\{(\theta_n, \mu_n)\}_{n \in \mathbb{N}} \subset M_2(\eta, C)\), \(\mu \in \mathcal{P}(K)\) and suppose that \(\{\mu_n\}_{n \in \mathbb{N}}\) converges to \(\mu\) in \(\mathcal{P}(K)\). Then there exists a metric \(\theta\) on \(K\) such that \((\theta, \mu) \in M_2(\eta, C)\) and some subsequence of \(\{\theta_n\}_{n \in \mathbb{N}}\) converges to \(\theta\) in \(C(K \times K)\).

**Proof.** Since \(M_2(\eta, C)\) is a compact subset of \(C(K \times K) \times \mathcal{P}(K)\) by Proposition 6.17, there exist \((\theta, \nu) \in M_2(\eta, C)\) and a subsequence \(\{(\theta_{nk}, \mu_{nk})\}_{k \in \mathbb{N}}\) of \(\{(\theta_n, \mu_n)\}_{n \in \mathbb{N}}\) converging to \((\theta, \nu)\) in \(C(K \times K) \times \mathcal{P}(K)\), but then \(\{\mu_{nk}\}_{k \in \mathbb{N}}\) converges to both \(\mu\) and \(\nu\), hence \(\mu = \nu\) and thus \((\theta, \mu) = (\theta, \nu) \in M_2(\eta, C)\). \(\square\)

We next observe that the set \(M_2(\eta, C)\) is almost invariant under the operation of pulling back by \(F_w\) followed by a suitable normalization, as stated in the following lemma.

**Lemma 6.19.** Let \((\theta, \mu) \in M_2(\eta, C)\), \(w \in W_\ast\) and define \((\theta_w, \mu_w) \in C(K \times K) \times \mathcal{P}(K)\) by

\[
\theta_w(x, y) := \frac{\theta(F_w(x), F_w(y))}{\sqrt{r_w \mu(K_w)}} \quad \text{and} \quad \mu_w(A) := \frac{\mu(F_w(A))}{\mu(K_w)}. \tag{6.36}
\]

Then \((\theta_w, \mu_w) \in M_2(c_{R_\epsilon}^{-1} \eta, C)\), where \(c_{R_\epsilon}\) is the constant in Lemma 6.11.

**Proof.** It is immediate from \((\theta, \mu) \in M_2(\eta, C)\) that \(\theta_w\) and \(\mu_w\) can be defined by (6.36) and are a metric and a Borel probability measure on \(K\), respectively, and that \((\theta_w, \mu_w)\) satisfies the inequalities in (6.28). Moreover, for any \(x, y, z \in K\) and \(t \in (0, \infty)\) with \(\theta_w(x, y) \leq t \theta(x, z)\), we have \(\theta(F_w(x), F_w(y)) \leq t \theta(F_w(x), F_w(z))\) and hence it follows from the \(\eta\)-quasisymmetry of \(\theta\) to \(R_\epsilon\) and Lemma 6.11 that

\[
c_{R_\epsilon} r_w R_\epsilon(x, y) \leq R_\epsilon(F_w(x), F_w(y)) \leq \eta(t) R_\epsilon(F_w(x), F_w(z)) \leq \eta(t) r_w R_\epsilon(x, z),
\]

so that \(R_\epsilon(x, y) \leq c_{R_\epsilon}^{-1} \eta(t) R_\epsilon(x, z)\), proving the \(c_{R_\epsilon}^{-1} \eta\)-quasisymmetry of \(\theta_w\) to \(R_\epsilon\). \(\square\)

The operation as in (6.36) of pulling back Borel measures on \(K\) by \(F_w\) is compatible with the analogous operation on \(\mathcal{F}\) (recall (6.6) and (6.7)) in the following sense.

**Lemma 6.20.** Let \(u \in \mathcal{F}\) and \(w \in W_\ast\). Then \(\Gamma(u, u)(F_w(A)) = r_w^{-1} \Gamma(u \circ F_w, u \circ F_w)(A)\) for any Borel subset \(A\) of \(K\), and in particular \(\Gamma(u, u)(K_w) = r_w^{-1} \mathcal{E}(u \circ F_w, u \circ F_w)\). Moreover, if \(\Gamma(u, u)(K_w) > 0\), then for any Borel subset \(A\) of \(K\),

\[
\frac{\Gamma(u, u)(F_w(A))}{\Gamma(u, u)(K_w)} = \Gamma(u_w, u_w)(A), \quad \text{where} \quad u_w := \mathcal{E}(u \circ F_w, u \circ F_w)^{-1/2} u \circ F_w. \tag{6.37}
\]

**Proof.** Since \(F_w : K \to K_w\) is a homeomorphism, the first assertion is easily seen to be equivalent to [HN, Lemma 4-(i)], and the second follows by choosing \(A = K\) in the first.
Furthermore if $\Gamma(u,u)(K_w) > 0$, then we see from the first and second assertions and the bilinearity of $\Gamma(f,g)$ in $f, g \in \mathcal{F}$ that for any Borel subset $A$ of $K$,

$$\frac{\Gamma(u,u)(F_w(A))}{\Gamma(u,u)(K_w)} = \frac{r_w^{-1}\Gamma(u \circ F_w, u \circ F_w)(A)}{r_w^{-1}\mathcal{E}(u \circ F_w, u \circ F_w)} = \Gamma(u_w, u_w)(A),$$

completing the proof. \qed

Recall that $\mu$ is a minimal energy-dominant measure of $(\mathcal{E}, \mathcal{F})$ for any $(\theta, \mu) \in \mathcal{M}_2$ by Proposition 6.10 and Proposition 2.11-(b), and hence in particular that $\Gamma(u,u)$ is absolutely continuous with respect to $\mu$ for any $(\theta, \mu) \in \mathcal{M}_2$ and any $u \in \mathcal{F}$. The following lemma is a special case of the well-known Lebesgue differentiation theorem.

**Lemma 6.21.** Let $(\theta, \mu) \in \mathcal{M}_2$, $u \in \mathcal{F}$ and set $f := d\Gamma(u,u)/d\mu$. Then $\mu$-a.e. $x \in K$ is an $(R_\mathcal{E}, \mu)$-Lebesgue point for $f$, i.e., satisfies

$$\lim_{s \downarrow 0} \frac{1}{\mu(B_{R_\mathcal{E}}(x,s))} \int_{B_{R_\mathcal{E}}(x,s)} |f(y) - f(x)| d\mu(y) = 0. \quad (6.38)$$

**Proof.** $(K, \theta, \mu)$ is VD by Proposition 6.10 and Theorem 4.4, which together with the quasisymmetry of $\theta$ to $R_\mathcal{E}$ and (3.1) implies that $(K, R_\mathcal{E}, \mu)$ is VD. Now since $C(K)$ is dense in $L^1(K, \mu)$ (see, e.g., [Rud, Theorem 3.14]) and $f \in L^1(K, \mu)$, the claim follows by Lebesgue’s differentiation theorem [Hei, (2.8)], which requires $(K, R_\mathcal{E}, \mu)$ to be VD. \qed

**Lemma 6.22.** Let $(\theta, \mu) \in \mathcal{M}_2$, $u \in \mathcal{F}$, let $f : K \to [0, \infty)$ be a Borel measurable $\mu$-version of $d\Gamma(u,u)/d\mu$ and let $x \in K$ satisfy (6.38). Then for any $\omega \in \pi^{-1}(x)$ and any $w \in W_s$,

$$\lim_{n \to \infty} \frac{\Gamma(u,u)(K_{[w],n}w)}{\mu(K_{[w],n}w)} = f(x). \quad (6.39)$$

**Proof.** Let $\omega \in \pi^{-1}(x)$, $w \in W_s$, $n \in \mathbb{N} \cup \{0\}$ and set $s_n := \text{diam}(K_{[w],n}R_\mathcal{E})$. Then by (6.21), (6.25), and VD of $(K, R_\mathcal{E}, \mu)$ noted in the above proof of Lemma 6.21, we have

$$\mu(K_{[w],n}w) \geq c[w]\mu(K_{[w],n}) \geq c[w]c'\mu(B_{R_\mathcal{E}}(x,2s_n)) \quad (6.40)$$

for some $c, c' \in (0, \infty)$ determined solely by $\mathcal{L}, (D, \mathbf{r}), (\theta, \mu)$. Now since $K_{[w],n}w \subset K_{[w],n} \subset B_{R_\mathcal{E}}(x,2s_n)$ and $\lim_{n \to \infty} s_n = 0$ by Lemma 6.11, it follows from (6.40) and (6.38) that

$$\left| \frac{\Gamma(u,u)(K_{[w],n}w)}{\mu(K_{[w],n}w)} - f(x) \right| = \left| \frac{1}{\mu(K_{[w],n}w)} \int_{K_{[w],n}w} (f(y) - f(x)) d\mu(y) \right|$$

$$\leq \frac{1}{\mu(K_{[w],n}w)} \int_{K_{[w],n}w} |f(y) - f(x)| d\mu(y)$$

$$\leq \frac{(c[w]c')^{-1}}{\mu(B_{R_\mathcal{E}}(x,2s_n))} \int_{B_{R_\mathcal{E}}(x,2s_n)} |f(y) - f(x)| d\mu(y) \xrightarrow{m \to \infty} 0,$$
On the other hand, since $\Gamma$ is bilinear, symmetric and non-negative definite, proving (6.39).

\[ \text{□} \]

Taking an $(R_\varepsilon, \mu)$-Lebesgue point $x \in K$ for $d\Gamma(u, u)/d\mu$ with $(d\Gamma(u, u)/d\mu)(x) > 0$ and considering the enlargements of infinitesimally small cells containing $x$ to the original scale as in Lemmas 6.19 and 6.20, we arrive at the following proposition.

**Proposition 6.23.** Let $(\theta, \mu) \in \mathcal{M}_2$, $u \in \mathcal{F}$, let $f : K \to [0, \infty)$ be a Borel measurable $\mu$-version of $d\Gamma(u, u)/d\mu$, let $x \in K$ satisfy (6.38) and $f(x) > 0$, and let $\omega \in \pi^{-1}(x)$. For each $n \in \mathbb{N} \cup \{0\}$, define $\mu_n := [\omega]_n \in \mathcal{P}(K)$ by (6.36) with $w = [\omega]_n$ and, noting that $\Gamma(u, u)(K[\omega]) > 0$ by (6.39), define $u_n := u_{[\omega]_n} \in \mathcal{F}$ by (6.37) with $w = [\omega]_n$. If $v \in \mathcal{F}$ and $\{u_n\}_{k \in \mathbb{N}} \subset \mathbb{N}$ is strictly increasing and satisfies $\lim_{k \to \infty} \mathcal{E}(v - u_{n_k}, v - u_{n_k}) = 0$, then $\Gamma(v, v) \in \mathcal{P}(K)$ and $\{\mu_{n_k}\}_{k \in \mathbb{N}}$ converges to $\Gamma(v, v)$ in $\mathcal{P}(K)$.

**Proof.** Let $w \in W_*$. Then we see from (6.36), (6.37) and (6.39) that

\[
\frac{\Gamma(u_n, u_n)(K_w)}{\mu_n(K_w)} = \left( \frac{\Gamma(u, u)(K[\omega]_n)}{\mu(K[\omega]_n)} \right)^{-1} \frac{\Gamma(u, u)(K[\omega]_{n^w})}{\mu(K[\omega]_{n^w})} \xrightarrow{m \to \infty} f(x)^{-1} f(x) = 1. \tag{6.41}
\]

On the other hand, since $\Gamma$ is bilinear, symmetric and non-negative definite,

\[
\left| \Gamma(v, v)(K_w)^{1/2} - \Gamma(u_{n_k}, u_{n_k})(K_w)^{1/2} \right|^2 \leq \Gamma(v - u_{n_k}, v - u_{n_k})(K_w) \leq \Gamma(v - u_{n_k}, v - u_{n_k})(K) = \mathcal{E}(v - u_{n_k}, v - u_{n_k}) \xrightarrow{k \to \infty} 0. \tag{6.42}
\]

It follows from (6.41) and (6.42) that

\[
\lim_{k \to \infty} \mu_{n_k}(K_w) = \Gamma(v, v)(K_w) \tag{6.43}
\]

and in particular that $\Gamma(v, v)(K) = \lim_{k \to \infty} \mu_{n_k}(K) = 1$, namely $\Gamma(v, v) \in \mathcal{P}(K)$. Note that $\mu_n(F_w(V_0)) = \mu(K[\omega]_n)^{-1} \mu(F_w(V_0)) = 0$ for any $n \in \mathbb{N} \cup \{0\}$ by Proposition 6.10 and that $\Gamma(v, v)(F_w(V_0)) = 0$ by $\#F_w(V_0) < \infty$ and [CF, Theorem 4.3.8], and recall that $K_w = F_w(K)$ is closed in $K$ and $K_w \setminus F_w(V_0)$ is open in $K$ as noted in the last paragraph of the proof of Proposition 6.17. By using these facts and the equality $K \setminus V_n = \bigcup_{v \in W_n} (K_v \setminus F_v(V_0))$, with the union disjoint, implied by [Kig01, Proposition 1.3.5-(2)] for any $n \in \mathbb{N}$, we easily see that the validity of (6.43) for any $w \in W_*$ is equivalent to the desired convergence of $\{\mu_{n_k}\}_{k \in \mathbb{N}}$ to $\Gamma(v, v)$ in $\mathcal{P}(K)$, which completes the proof. \[ \text{□} \]

Now we can conclude the proof of the main result of this subsection (Theorem 6.16).

**Proof.** [Proof of Theorem 6.16] By the assumption $\mathcal{M}_2(\eta, C) \neq \emptyset$ we can take $(\theta, \mu) \in \mathcal{M}_2(\eta, C)$. Let $u \in \mathcal{H}_0 \setminus \mathbb{R}^*_1 K$, which exists by Proposition 6.6-(1) and $\#V_0 \geq 2$, and let $f : K \to [0, \infty)$ be a Borel measurable $\mu$-version of $d\Gamma(u, u)/d\mu$. Then since $\mu(f^{-1}((0, \infty))) > 0$ by $\int_K f \, d\mu = \Gamma(u, u)(K) = \mathcal{E}(u, u) > 0$, Lemma 6.21 implies that there exists $x \in K$ with the properties (6.38) and $f(x) > 0$. Let $\omega \in \pi^{-1}(x)$, and for each $n \in \mathbb{N} \cup \{0\}$, as in Proposition 6.23 define $(\theta_n, \mu_n) := ([\omega]_n, \mu_{[\omega]_n}) \in C(K \times K) \times \mathcal{P}(K)$ by (6.36)
and \( u_n := u_{[w]} \in \mathcal{F} \) by (6.37) with \( w = [\omega] \), so that \( \{(\theta_n, \mu_n)\}_{n \in \mathbb{N} \cup \{0\}} \subset \mathcal{M}_2(c_{\mathbb{R}_+}^{-1} \eta, C) \) by Lemma 6.19 and \( \{u_n\}_{n \in \mathbb{N} \cup \{0\}} \subset \{h \in \mathcal{H}_0 \mid \mathcal{E}(h, h) = 1\} \) by Proposition 6.6-(2). Noting that \( \mathcal{H}_0/\mathbb{R}_{1K} \) is a finite dimensional linear space with inner product \( \mathcal{E} \) by (RF1) and hence that \( \{h \in \mathcal{H}_0/\mathbb{R}_{1K} \mid \mathcal{E}(h, h) = 1\} \) is a compact subset of \( \mathcal{H}_0/\mathbb{R}_{1K} \), we get \( h \in \mathcal{H}_0 \setminus \mathbb{R}_{1K} \) and a strictly increasing sequence \( \{n_k\}_{k \in \mathbb{N}} \subset \mathbb{N} \) satisfying \( \mathcal{E}(h, h) = 1 \) and \( \lim_{k \to \infty} \mathcal{E}(h - u_{n_k}, h - u_{n_k}) = 0 \). Then \( \Gamma(h, h) \in \mathcal{P}(K) \) and \( \{\mu_{n_k}\}_{k \in \mathbb{N}} \) converges to \( \Gamma(h, h) \) in \( \mathcal{P}(K) \) by Proposition 6.23, and it follows from \( \{(\theta_{n_k}, \mu_{n_k})\}_{k \in \mathbb{N}} \subset \mathcal{M}_2(c_{\mathbb{R}_+}^{-1} \eta, C) \) and Corollary 6.18 that \( (\theta_h, \Gamma(h, h)) \in \mathcal{M}_2(c_{\mathbb{R}_+}^{-1} \eta, C) \) for some metric \( \theta_h \) on \( K \).

\[ \square \]

### 6.3 Examples

In this subsection, we show that the value \( d_{cw} = 2 \) of the conformal walk dimension fails to be attained for some concrete examples of post-critically finite self-similar sets.

#### 6.3.1 The Vicsek set

**Example 6.24** (Vicsek set). Set \( S := \{0, 1, 2, 3, 4\} \), define \( \{q_i\}_{i \in S} \subset \mathbb{R}^2 \) by \( q_0 := (0, 0) \), \( q_1 := (1, 1) \), \( q_2 := (-1, 1) \), \( q_3 := (-1, -1) \) and \( q_4 := (1, -1) \), and define \( f_i : \mathbb{R}^2 \to \mathbb{R}^2 \) for each \( i \in S \) by \( f_i(x) := q_i + \frac{1}{2}(x - q_i) \). Let \( K \) be the self-similar set associated with \( \{f_i\}_{i \in S} \), i.e., the unique non-empty compact subset of \( \mathbb{R}^2 \) such that \( K = \bigcup_{i \in S} f_i(K) \), which exists and satisfies \( K \subset [0, 1]^2 \) thanks to \( \bigcup_{i \in S} f_i([-1, 1]^2) \subset [-1, 1]^2 \) by [Kig01, Theorem 1.1.4], and set \( F_i := f_i|_K \) for each \( i \in S \). Then \( \mathcal{L} := (K, S, \{F_i\}_{i \in S}) \) is a self-similar structure by [Kig01, Theorem 1.2.3] and called the Vicsek set (see Figure 1 below), and it easily follows from \( K \subset [-1, 1]^2 \) that \( \mathcal{P} = \{1^\infty, 2^\infty, 3^\infty, 4^\infty\} \) and \( V_0 = \{q_1, q_2, q_3, q_4\} \), so that \( \mathcal{L} \) is post-critically finite.

Let \( r = \left(0, \frac{1}{2}\right) \), set \( \mathbf{r} = (r_i)_{i \in S} := (-2r, r, r, r, r) \), and define \( D = (D_{pq})_{p,q \in V_0} \) and \( D' = (D'_{pq})_{p,q \in V_0 \cup \{\varnothing\}} \) by

\[
D_{pq} := \begin{cases} 
1 & \text{if } p \neq q, \\
-3 & \text{if } p = q,
\end{cases} \quad D'_{pq} := \begin{cases} 
4 & \text{if } \{p, q\} = \{q_0, q_i\} \text{ for some } i \in S \setminus \{0\}, \\
0 & \text{if } \{p, q\} = \{q_i, q_j\} \text{ for some } i, j \in S \setminus \{0\}, \\
-16 & \text{if } p = q = q_0, \\
-4 & \text{if } p = q = q_i \text{ for some } i \in S \setminus \{0\}.
\end{cases}
\]

Then setting \( \mathcal{E}^{(0)}(u, v) := -\sum_{p,q \in V_0 \cup \{\varnothing\}} D'_{pq} u(q) v(p) \) for \( u, v \in \mathbb{R}^{V_0 \cup \{\varnothing\}} \), we immediately see that \( \mathcal{E}^{(0)}(u, u) = \inf_{v \in \mathbb{R}^{V_0 \cup \{\varnothing\}}} \mathcal{E}^{(0)}(v, v) \) for any \( u \in \mathbb{R}^{V_0 \cup \{\varnothing\}} \), which in turn easily implies that \( (D, \mathbf{r}) \) is a regular harmonic structure on \( \mathcal{L} \).

In the situation of Example 6.24, we have the following proposition:

**Proposition 6.25.** Let \( \mathcal{L} = (K, S, \{F_i\}_{i \in S}) \) and \( \mathcal{L} = (D, \mathbf{r}) \) be as in Example 6.24 and set \( I_{i, i+2} := \{(1 - t)q_i + t q_{i+2} \mid t \in [0, 1]\} \) for \( i \in \{1, 2\} \). Then \( \Gamma(h, h) \left( K \setminus (I_{1.3} \cup I_{2.4}) \right) = 0 \) for any \( h \in \mathcal{H}_0 \).
Proof. Let $U$ be a connected component of $K \setminus (I_{1,3} \cup I_{2,4})$. Then it is immediate from $K = \bigcup_{i \in S} F_i(K) \subseteq [-1,1]^2$ that $U \cap (I_{1,3} \cup I_{2,4})$ consists of a unique element $q_U$. Therefore for any $u \in \mathcal{F}$, the function $u_U \in C(K)$ defined by $u_U|_U := u(q_U)1_U$ and $u_U|_{K \setminus U} := u|_{K \setminus U}$ is easily seen to satisfy $u_U \in \mathcal{F}$ and $\mathcal{E}(u,U) \geq \mathcal{E}(u_U,u_U)$, with the equality holding if and only if $u = u_U$. In particular, any $u \in \mathcal{F}$ with $u \neq u_U$ fails to be 0-harmonic by $U \cap V_0 = \emptyset$, so that any $h \in \mathcal{H}_0$ has to be constant on $U$, hence $\Gamma(h, h)(U) = 0$ and thus $\Gamma(h, h)(K \setminus (I_{1,3} \cup I_{2,4})) = 0$ since $U$ is an arbitrary connected component of $K \setminus (I_{1,3} \cup I_{2,4})$ (and there are only countably many of them).

Corollary 6.26. Let $\mathcal{L} = (K, S, \{F_i\}_{i \in S})$ and $(D, r)$ be as in Example 6.24. Then $\mathcal{M}_2(\mathcal{L}, D, r) = \emptyset$, i.e., the value $d_{cw} = 2$ of the conformal walk dimension is NOT attained for the MMD space $(K, R_\xi, m, \mathcal{E}, \mathcal{F})$.

Proof. If $\mathcal{M}_2 = \mathcal{M}_2(\mathcal{L}, D, r) \neq \emptyset$, then $\mathcal{M}_2(\eta, C) \neq \emptyset$ for some $\eta \in \text{Homeo}^+$ and $C \in (1, \infty)$, and Theorem 6.16 would imply that $(d_h, \Gamma(h, h)) \in \mathcal{M}_2(c_{R_\xi}^{-1}\eta, C)$ for some $h \in \mathcal{H}_0 \setminus \mathbb{R} \setminus \mathcal{N}$ and some metric $d_h$ on $K$, but then $\Gamma(h, h)(K_{12}) \geq C^{-1}r_{12}^{-1}(\text{diam}(K_{12}, d_h))^2 > 0$, which would contradict Proposition 6.25 since $K_{12} \subset K \setminus (I_{1,3} \cup I_{2,4})$. Thus $\mathcal{M}_2 = \emptyset$. □

6.3.2 Higher-dimensional Sierpiński gaskets

Example 6.27. ($N$-dimensional Sierpiński gasket). Let $N \in \mathbb{N}$, $N \geq 2$ and let $\{q_k\}_{k=0}^N \subset \mathbb{R}^N$ be the set of the vertices of a regular $N$-dimensional simplex $\Delta_N$. We further set $S := \{0, 1, \ldots, N\}$ and for each $i \in S$ define $f_i : \mathbb{R}^N \to \mathbb{R}^N$ by $f_i(x) := q_i + \frac{1}{2}(x - q_i)$. Let $K$ be the self-similar set associated with $\{f_i\}_{i \in S}$, which exists and satisfies $K \subseteq \Delta_N$ thanks to $\bigcup_{i \in S} f_i(\Delta_N) \subseteq \Delta_N$ by [Kig01, Theorem 1.1.4], and set $F_i := f_i|_K$ for each $i \in S$. Then $\mathcal{L} := (K, S, \{F_i\}_{i \in S})$ is a self-similar structure by [Kig01, Theorem 1.2.3] and called the $d$-dimensional (standard) Sierpiński gasket (see Figure 1 above), and it easily follows from $K \subset \Delta_N$ that $\mathcal{P} = \{k^{\infty} \mid k \in S\}$ and $V_0 = \{q_k \mid k \in S\}$, so that $\mathcal{L}$ is post-critically finite.

Define $D = (D_{pq})_{p,q \in V_0}$ by $D_{pp} := -N$ and $D_{pq} := 1$ for $p, q \in V_0$ with $p \neq q$. By the symmetry of $\mathcal{L}$ and $D$, there exists a unique $r \in (0, \infty)$ such that $(D, r) = (r_i)_{i \in S}$ with $r_i := r$ is a harmonic structure on $\mathcal{L}$, and an elementary calculation shows that $r = \frac{N+1}{N+3} < 1$, so that $(D, r)$ is a regular harmonic structure on $\mathcal{L}$. 

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We fix the setting of Example 6.27 in the rest of this subsection. The following result, which is essentially due to Kigami [Kig08], is the starting point of the whole study of the present paper:

**Theorem 6.28** ([Kig08, Kaj12]). Assume that $N = 2$, let $h_1, h_2 \in \mathcal{H}_0$ satisfy $\mathcal{E}(h_1, h_1) + \mathcal{E}(h_2, h_2) = 1$, and set $\mu_{h_1, h_2} := \Gamma(h_1, h_1) + \Gamma(h_2, h_2)$. Then there exists a metric $d_{h_1, h_2}$ on $K$ such that $(d_{h_1, h_2}, \mu_{h_1, h_2}) \in \mathcal{M}_2$. In particular, the value $d_{cw} = 2$ is attained for the MMD space $(K, R, m, \mathcal{E}, \mathcal{F})$.

One of the key observations for the validity of Theorem 6.28 is that the energy measures of harmonic functions are volume doubling with respect to the resistance metric $R$ (or equivalently, with respect to the Euclidean metric), which in fact extends to the $N$-dimensional Sierpiński gasket with $N \geq 3$ as follows:

**Proposition 6.29.** $(K, R, \Gamma(h, h))$ is VD for any $h \in \mathcal{H}_0 \setminus \mathbb{R}1_K$.

**Proof.** This can be proved in essentially the same way as [Kig08, Proof of Theorem 3.2]. □

Despite Proposition 6.29, Theorem 6.28 does NOT extend to the case of $N \geq 3$, which is the main result of this subsection and stated as follows:

**Theorem 6.30.** Assume that $N \geq 3$, and let $h \in \mathcal{H}_0$ satisfy $\mathcal{E}(h, h) = 1$. Then there does NOT exist a metric $d_h$ on $K$ such that $(d_h, \Gamma(h, h)) \in \mathcal{M}_2$.

**Corollary 6.31.** Assume that $N \geq 3$. Then Then $\mathcal{M}_2(\mathcal{L}, D, r) = \emptyset$, i.e., the value $d_{cw} = 2$ of the conformal walk dimension is NOT attained for the MMD space $(K, R, m, \mathcal{E}, \mathcal{F})$.

**Proof.** If $\mathcal{M}_2 = \mathcal{M}_2(\mathcal{L}, D, r) \neq \emptyset$, then $\mathcal{M}_2(\eta, C) \neq \emptyset$ for some $\eta \in \text{Homeo}^+$ and $C \in (1, \infty)$, and Theorem 6.16 would imply that $(d_h, \Gamma(h, h)) \in \mathcal{M}_2(c_{R}^{-1}\eta, C) \subset \mathcal{M}_2$ for some $h \in \mathcal{H}_0 \setminus \mathbb{R}1_K$ and some metric $d_h$ on $K$, which would contradict Theorem 6.30 and thereby proves that $\mathcal{M}_2 = \emptyset$. □

**Proof.** [Proof of Theorem 6.30] Let $h \in \mathcal{H}_0$ satisfy $\mathcal{E}(h, h) = 1$. By the symmetry of $\mathcal{L}$ and $D$, we may assume that $h(q_0) = \max_{q \in V_0} h(q)$. Let $\psi \in \mathcal{H}_0$ satisfy $\psi|_{V_0} = N^{-1/2}1_{\psi(q_0)}$. Then it is elementary to see that $F_0^*\psi = \frac{N + 1}{N + 3}\psi$ and that by $h(q_0) = \max_{q \in V_0} h(q)$ there exist $a, c \in \mathbb{R}$ with $a > 0$ such that $h - a\psi - c1_K$ is an eigenfunction of $F_0^*: \mathcal{H}_0 \rightarrow \mathcal{H}_0$ with eigenvalue $\frac{1}{N + 3}$. We thus obtain

$$\lim_{n \rightarrow \infty} \mathcal{E}(\psi - h_{0^n}, \psi - h_{0^n}) = 0, \quad \text{where } h_{0^n} := \mathcal{E}(h \circ F_{0^n}, h \circ F_{0^n})^{-1/2}h \circ F_{0^n} \text{ for } n \in \mathbb{N}. \tag{6.44}$$

Now suppose that there were a metric on $K$ such that $(d_h, \Gamma(h, h)) \in \mathcal{M}_2$, and let $\eta \in \text{Homeo}^+$ and $C \in (1, \infty)$ be such that $(d_h, \Gamma(h, h)) \in \mathcal{M}_2(\eta, C)$. For each $n \in \mathbb{N}$, define $d_{h_{0^n}}$ by (6.36) with $d = d_h$ and $w = 0^n$, so that $\{(d_{h_{0^n}}, \Gamma(h_{0^n}, h_{0^n}))\}_{n \in \mathbb{N}} \subset \mathcal{M}_2(c_{R}^{-1}\eta, C)$ by Lemmas 6.20 and 6.19. Then since $\{\Gamma(h_{0^n}, h_{0^n})\}_{n \in \mathbb{N}}$ converges to $\Gamma(\psi, \psi)$
in \( \mathcal{P}(K) \) by (6.44), Corollary 6.18 would imply that \((d_\varphi, \Gamma(\psi, \psi)) \in \mathcal{M}_2(c_{R_2}^1, C)\) for some metric \(d_\varphi\) on \(K\). Next, set \(\varphi := \mathcal{E}(\psi \circ F_1, \psi \circ F_1)^{-1/2} \circ F_1 - N^{-1/2}(N + 3)^{-1} \mathbb{1}_K\) and define \(d_\varphi : K \times K \to [0, \infty)\) by \(d_\varphi(x, y) := (r_1 \Gamma(\psi, \psi)(K_1))^{-1/2} d_\psi(F_1(x), F_1(y))\), so that \((C, \Gamma(\varphi, \varphi)) \in \mathcal{M}_2(c_{R_2}^1, C)\) by Lemmas 6.20 and 6.19. In particular, it follows from Proposition 6.10 and Proposition 2.11 that the intrinsic metric \(d_{\text{int}, \varphi}\) of the Dirichlet space \((K, \Gamma(\varphi, \varphi), \mathcal{E}, \mathcal{F})\) would be bi-Lipschitz equivalent to \(d_\varphi\) and hence would be itself a metric on \(K\).

On the other hand, an explicit calculation of which in turn yields \(\varphi \circ F_j = (N + 3)^{-1} \varphi\) for any \(j \in S \setminus \{0, 1\}\). Now let \(u \in \mathcal{F}\) satisfy \(\Gamma(u, u) \leq \Gamma(\varphi, \varphi)\). Then for any \(w \in \bigcup_{n=1}^\infty (S \setminus \{0, 1\})^n\) and any \(x, y \in K_w\), we see from Lemma 6.20 that

\[
|u(x) - u(y)|^2 = |(u \circ F_w)(F_w^{-1}(x)) - (u \circ F_w)(F_w^{-1}(y))|^2 \\
\leq \text{diam}(K, R_\mathcal{E}) \mathcal{E}(u \circ F_w, u \circ F_w) = \text{diam}(K, R_\mathcal{E}) r_w \Gamma(u, u)(K_w) \\
\leq \text{diam}(K, R_\mathcal{E}) r_w \Gamma(\varphi, \varphi)(K_w) = \text{diam}(K, R_\mathcal{E}) \mathcal{E}(\varphi \circ F_w, \varphi \circ F_w) \\
= \text{diam}(K, R_\mathcal{E})(N + 3)^{-2} |w| \mathcal{E}(\varphi, \varphi). \tag{6.45}
\]

For any \(n \in \mathbb{N}\), using (6.45) with \(w \in \{2, 3\}^n\), \(x = F_w(q_2)\) and \(y = F_w(q_3)\), we now obtain

\[
|u(q_2) - u(q_3)| \leq \sum_{w \in \{2, 3\}^n} |u(F_w(q_2)) - u(F_w(q_3))| \\
\leq 2^n (\text{diam}(K, R_\mathcal{E}) \mathcal{E}(\varphi, \varphi))^{1/2} (N + 3)^{-n} \xrightarrow{n \to \infty} 0
\]

and thus \(u(q_2) - u(q_3) = 0\). Since \(u \in \mathcal{F}\) satisfies \(\Gamma(u, u) \leq \Gamma(\varphi, \varphi)\) is arbitrary, it follows that the intrinsic metric \(d_{\text{int}, \varphi}\) of the Dirichlet space \((K, \Gamma(\varphi, \varphi), \mathcal{E}, \mathcal{F})\) satisfies \(d_{\text{int}, \varphi}(q_2, q_3) = 0\), which would contradict the conclusion of the previous paragraph that \(d_{\text{int}, \varphi}\) would be a metric on \(K\), completing the proof. \(\square\)

### 6.4 The case of generalized Sierpiński carpets

In this subsection, we treat the case of the canonical self-similar Dirichlet form on an arbitrary generalized Sierpiński carpet and see that the arguments in Subsection 6.2 go through also in this case with just slight modifications required in the proofs.

We fix the following setting throughout this subsection.

**Framework 6.32.** Let \(N, l \in \mathbb{N}\), \(N \geq 2\), \(l \geq 3\) and set \(Q_0 := [0, 1]^N\). Let \(S \subset \{0, 1, \ldots, l - 1\}^N\) be non-empty, define \(f_i : \mathbb{R}^N \to \mathbb{R}^N\) by \(f_i(x) := l^{-1} i + l^{-1} x\) for each \(i \in S\) and set \(Q_1 := \bigcup_{i \in S} f_i(Q_0)\), so that \(Q_1 \subsetneq Q_0\). Let \(K\) be the self-similar set associated with \(\{f_i\}_{i \in S}\), i.e., the unique non-empty compact subset of \(\mathbb{R}^N\) such that \(K = \bigcup_{i \in S} f_i(K)\), which exists and satisfies \(K \subsetneq Q_0\) thanks to \(Q_1 \subsetneq Q_0\) by [Kig01, Theorem 1.1.4], and set \(F_i := f_{i|K}\) for each \(i \in S\), so that \(\text{GSC}(N, l, S) := (K, S, \{F_i\}_{i \in S})\) is a self-similar structure by [Kig01, Theorem 1.2.3]. Let \(d : K \times K \to [0, \infty)\) be the Euclidean metric.
on $K$ given by $d(x, y) := |x - y|$, set $d_I := \log \#S$, and let $m$ be the self-similar measure on $\text{GSC}(N, l, S)$ with weight $(1/\#S)_{i \in S}$, i.e., the unique Borel probability measure on $K$ such that $m(K_w) = (\#S)^{-|w|}$ for any $w \in \mathcal{W}_*$, which exists by [Kig01, Propositions 1.5.8, 1.4.3, 1.4.4 and Corollary 1.4.8].

Recall that $d_I$ is the Hausdorff dimension of $(K, d)$ and that $m$ is a constant multiple of the $d_I$-dimensional Hausdorff measure on $(K, d)$; see, e.g., [Kig01, Proposition 1.5.8 and Theorem 1.5.7]. Note that $d_I < N$ by $S \subset \{0, 1, \ldots, l-1\}^N$.

The following definition is essentially due to Barlow and Bass [BB99, Section 2].

**Definition 6.33** (Generalized Sierpiński carpet, [BBKT, Subsection 2.2]). $\text{GSC}(N, l, S)$ is called a **generalized Sierpiński carpet** if and only if the following four conditions are satisfied:

1. **(GSC1) (Symmetry)** $f(Q_1) = Q_1$ for any isometry $f$ of $\mathbb{R}^N$ with $f(Q_0) = Q_0$.
2. **(GSC2) (Connectedness)** $Q_1$ is connected.
3. **(GSC3) (Non-diagonality)** $\text{int}_{\mathbb{R}^N}(Q_1 \cap \prod_{k=1}^N [(i_k - \varepsilon_k)l^{-1}, (i_k + 1)l^{-1}]$ is either empty or connected for any $(i_k)_{k=1}^N \in \mathbb{Z}^N$ and any $(\varepsilon_k)_{k=1}^N \in \{0, 1\}^N$.
4. **(GSC4) (Borders included)** $[0, 1] \times \{0\}^{N-1} \subset Q_1$.

As special cases of Definition 6.33, $\text{GSC}(2, 3, S_{SC})$ and $\text{GSC}(3, 3, S_{MS})$ are called the **Sierpiński carpet** and the **Menger sponge**, respectively, where $S_{SC} := \mathcal{P}^2 \setminus \{(1, 1)\}$ and $S_{MS} := \{(i_1, i_2, i_3) \in \{0, 1, 2\}^3 \mid \sum_{k=1}^3 \mathbb{1}_{\{1\}}(i_k) \leq 1\}$ (see Figure 2 below).

See [BB99, Remark 2.2] for a description of the meaning of each of the four conditions (GSC1), (GSC2), (GSC3) and (GSC4) in Definition 6.33. We remark that there are several equivalent ways of stating the non-diagonality condition, as in the following proposition.

**Proposition 6.34** ([Kaj10, §2]). Set $|x|_1 := \sum_{k=1}^N |x_k|$ for $x = (x_k)_{k=1}^N \in \mathbb{R}^N$. Then (GSC3) is equivalent to any one of the following three conditions:

1. **(ND)** $\text{int}_{\mathbb{R}^N}(Q_1 \cap \prod_{k=1}^N [(i_k - 1)l^{-n}, (i_k + 1)l^{-n}]$ is either empty or connected for any $n \in \mathbb{N}$ and any $(i_k)_{k=1}^N \in \{1, 2, \ldots, l^n - 1\}^N$.

Figure 2: Sierpiński carpet, some other generalized Sierpiński carpets with $N = 2$ and Menger sponge
There are two established ways of constructing a non-degenerate Dirichlet form. (Theorems 1.2 and 4.32), (Hin13, Proposition 5.1), (Kaj13, Proposition 5.9). There exists a unique (up to constant multiples of $\mathcal{E}$) regular symmetric Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K,m)$ satisfying $\mathcal{E}(u,u) > 0$ for some $u \in \mathcal{F}$, $\mathcal{E}(1_K, 1_K) = 0$, (6.6), (6.7) for some $r \in (0, \infty)$, and the following:

$\mathcal{E}$-symmetry

$$\mathcal{E}(u \circ g, u \circ g) = \mathcal{E}(u, u).$$
**Definition 6.40.** The regular symmetric Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K, m)$ as in Theorem 6.39 is called the canonical Dirichlet form on GSC($N, I, S$), and we set $d_w := \log(\#S/r)$. Note that $(\mathcal{E}, \mathcal{F})$ is also strongly local by the same argument as [Hin05, Proof of Lemma 3.12] based on (6.6), (6.7) and $\mathcal{E}(1_K, 1_K) = 0$.

**Theorem 6.41** ([BB99, Remarks 5.4-1.], [Kaj20, Theorem 2.9]). $d_w > 2$.

**Theorem 6.42** ([BB99, Theorem 1.3], [BBKT, Theorem 4.30 and Remark 4.33]). There exists a (unique) continuous version $p = p_t(x, y) : (0, \infty) \times K \times K \to [0, \infty)$ of the heat kernel of $(K, d, m, \mathcal{E}, \mathcal{F})$, and there exist $c_1, c_2, c_3, c_4 \in (0, \infty)$ such that for any $(t, x, y) \in (0, 1] \times K \times K$,

$$
\frac{c_1}{t^{d/d_w}} \exp\left(-\left(\frac{d(x, y)^{d_w}}{c_2 t}\right)^{\frac{1}{\gamma}}\right) \leq p_t(x, y) \leq \frac{c_3}{t^{d/d_w}} \exp\left(-\left(\frac{d(x, y)^{d_w}}{c_4 t}\right)^{\frac{1}{\gamma}}\right). \tag{6.47}
$$

In particular, $(K, d, m, \mathcal{E}, \mathcal{F})$ satisfies $\text{HKE}(d_w)$ and $\text{PHI}(d_w)$ by Lemma 6.37 and Theorem 4.4.

As the counterpart of Proposition 6.10 for generalized Sierpiński carpets, we have the following characterization of the pair $(\theta, \mu)$ of $\theta \in \mathcal{J}(K, d)$ and $\mu \in \mathcal{A}(K, d, m, \mathcal{E}, \mathcal{F})$ such that $(K, \theta, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu)$ satisfies $\text{PHI}(\beta)$; see also [Kig19] for related results.

**Theorem 6.43.** Let $\theta$ be a metric on $K$ quasisymmetric to $d$, let $\mu$ be a Radon measure on $K$ with full support and let $\beta \in [2, \infty)$. Then the following conditions are equivalent:

1. $\mu$ is admissible with respect to $(K, d, m, \mathcal{E}, \mathcal{F})$ and $(K, \theta, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu)$ satisfies $\text{PHI}(\beta)$.

2. There exists $C \in (1, \infty)$ such that

$$
C^{-1}(\text{diam}(K_w, \theta))^{\beta} \leq r_w \mu(K_w) \leq C(\text{diam}(K_w, \theta))^{\beta} \quad \text{for any } w \in W, \tag{6.48}
$$

where $r_w := r^{\text{vol}_w}$. Moreover, if either of these conditions holds, then $\mu(F_w(V_0)) = 0$ for any $w \in W$ and $\mu(\{x\}) = 0$ for any $x \in K$.

Theorem 6.43 follows by repeating the same arguments as the proof of Proposition 6.10, on the basis of the following proposition concluded from Theorem 4.4 together with [BCM, Lemma 5.21 and Proposition 6.17].

**Proposition 6.44.** Let $\theta$ be a metric on $K$ quasisymmetric to $d$, let $\mu$ be a Radon measure on $K$ with full support and let $\beta \in [2, \infty)$. Then Theorem 6.43-(a) is equivalent to the following condition:

1. $(K, \theta, \mu)$ is VD and there exists $C \in (0, \infty)$ such that for any $x, y \in K$ with $x \neq y$,

$$
C^{-1}(\theta(x, y))^{\beta} \leq d(x, y)^{d_w - d_t} \mu(B_{\theta}(x, \theta(x, y))) \leq C(\theta(x, y))^{\beta}. \tag{6.49}
$$
Proof.  Note first that, if $\mu$ is admissible with respect to $(K, d, m, \mathcal{E}, \mathcal{F})$, then for any Borel subsets $A, B$ of $K$ with $B$ closed in $K$ and $\overline{A} \cap B = \emptyset$, by [CF, Theorem 5.2.11] and [FOT, Theorem 4.6.2 and Lemma 2.1.4] we have

$$\text{Cap}^\mu(A, B) = \text{cap}(A, B).$$

where $\text{Cap}^\mu(A, B)$ and $\text{Cap}(A, B)$ denote the capacity between $A, B$ with respect to $(K, \theta, \mu, \mathcal{E}, \mathcal{F})$ and $(K, d, m, \mathcal{E}, \mathcal{F})$, respectively. Next, since $(K, d, m, \mathcal{E}, \mathcal{F})$ satisfies (3.1) and (3.3), there exist $A_1, A_2 \in (1, \infty)$ with $A_2 \geq 2$ such that for any $(x, s) \in K \times (0, \text{diam}(K, d)/A_2)$,

$$C_1^{-1} s^{d - d_w} \leq \text{Cap}(B_d(x, s), K \setminus B_d(x, A_1 s)) \leq C_1 s^{d - d_w},$$

and by virtue of the quasisymmetry of $\theta$ to $d$ and (3.1) we have EHI for $(K, \theta, m, \mathcal{E}, \mathcal{F})$, as well as for $(K, \theta, \mu, \mathcal{E}, \mathcal{F})$ provided $\mu$ is admissible with respect to $(K, d, m, \mathcal{E}, \mathcal{F})$.

To prove the desired equivalence, assume Theorem 6.43-(a), so that $(K, \theta, \mu, \mathcal{E}, \mathcal{F})$ satisfies VD and $\text{cap}(\beta)$ by Theorem 4.4 and therefore in view of (6.50) there exist $C_2, A_3, A_4 \in (1, \infty)$ such that

$$C_2^{-1} \frac{\mu(B_\theta(x, s))}{s^3} \leq \text{Cap}(B_\theta(x, s), K \setminus B_\theta(x, A_3 s)) \leq C_2 \frac{\mu(B_\theta(x, s))}{s^3}$$

for any $(x, s) \in K \times (0, \text{diam}(K, \theta)/A_4)$. To verify (6.49), let $x, y \in K$ satisfy $x \neq y$. By the quasisymmetry of $\theta$ to $d$, (3.1) and (3.3), there exist $A_5 \in (1, \infty)$ determined solely by $d, \theta, A_1$ and $A_6 \in (1, \infty)$ determined solely by $d, \theta, A_3$ such that

$$B_\theta(x, \theta(x, y)/A_5) \subset B_d(x, d(x, y)) \subset B_d(x, A_1 d(x, y)) \subset B_\theta(x, A_5 \theta(x, y)),$$

and

$$B_d(x, d(x, y)/A_6) \subset B_\theta(x, \theta(x, y)) \subset B_\theta(x, A_3 \theta(x, y)) \subset B_d(x, A_6 d(x, y)).$$

Then by EHI for $(K, d, m, \mathcal{E}, \mathcal{F})$ and $(K, \theta, m, \mathcal{E}, \mathcal{F})$ and [BCM, Lemma 5.21] (note also [BCM, Theorem 5.4 and Lemma 5.2-(e)]), there exist $A_7 \in (A_2, \infty)$ and $A_8 \in (A_4, \infty)$ such that for any $(z, s) \in K \times (0, \infty)$,

$$\text{Cap}(B_d(z, s), K \setminus B_d(z, A_1 s)) \asymp \text{Cap}(B_d(z, s), K \setminus B_d(z, A_5^2 s)) \quad \text{if } s < \text{diam}(K, d)/A_7,$$

and

$$\text{Cap}(B_\theta(z, s), K \setminus B_\theta(z, A_3 s)) \asymp \text{Cap}(B_\theta(z, s), K \setminus B_\theta(z, A_5^2 s)) \quad \text{if } s < \text{diam}(K, \theta)/A_8.$$  

(To be precise, the definition of capacity between sets in [BCM, Section 5] is slightly different from ours, but they are easily seen to be equivalent to each other by virtue of [FOT, Lemma 2.2.7-(ii)].) Moreover, the quasisymmetry of $\theta$ to $d$ again and (3.5) show
that by taking $A_8$ large enough we may further assume that $\theta(x, y) < \text{diam}(K, \theta)/A_8$ implies $d(x, y) < \text{diam}(K, d)/A_7$. Now, if $\theta(x, y) \geq \text{diam}(K, \theta)/A_8$, then (6.49) clearly holds for some sufficiently large $C \in (0, \infty)$ independent of $x, y$ since $\mu(B_\theta(x, \theta(x, y))) \asymp \mu(K)$ by VD of $(K, \theta, \mu)$ and $d(x, y) \asymp \text{diam}(K, d)$ by the quasisymmetry of $\theta$ to $d$ and (3.5). Otherwise $\theta(x, y) < \text{diam}(K, \theta)/A_8$, which implies $d(x, y) < \text{diam}(K, d)/A_7$, hence

$$C_2 \frac{\mu(B_\theta(x, \theta(x, y)))}{\theta(x, y)^\beta} \geq \text{Cap}(B_\theta(x, \theta(x, y)), K \setminus B_\theta(x, A_3 \theta(x, y)))$$

$$\geq \text{Cap}(B_d(x, d(x, y)/A_6), K \setminus B_d(x, A_6 d(x, y))) \asymp d(x, y)^{d_\ell - d_w}$$

by (6.52), (6.54), (6.55) and (6.51), and similarly

$$C_1 d(x, y)^{d_\ell - d_w} \geq \text{Cap}(B_d(x, d(x, y)), K \setminus B_d(x, A_1 d(x, y)))$$

$$\geq \text{Cap}(B_\theta(x, \theta(x, y)/A_5), K \setminus B_\theta(x, A_5 \theta(x, y))) \asymp \frac{\mu(B_\theta(x, \theta(x, y)))}{\theta(x, y)^\beta}$$

by (6.51), (6.53), (6.56), (6.52) and VD of $(K, \theta, \mu)$, proving (6.49) and thereby (c).

Conversely, assume (c). Since $K$ is connected, VD of $(K, \theta, \mu)$ implies RVD of $(K, \theta, \mu)$ by Remark 3.18-(b). Note that (6.53) remains valid and that for each $A_3, A_4 \in (1, \infty)$ we still have (6.54), (6.55) and (6.56). Note also that by the connectedness of $K$ and $[\text{Hei}, \text{Theorem 11.3}]$ there exist $\lambda, \alpha \in [1, \infty)$ such that $\theta$ is $\eta_{\alpha, \lambda}$-quasisymmetric to $d$ with $\eta_{\alpha, \lambda}$ as in Definition 3.1. Let $x \in K$ and let $s_1, s_2 \in (0, \text{diam}(K, d)/A_2)$ satisfy $s_1 \leq s_2$. Then $d(x, y) = s_1$ and $d(x, z) = s_2$ for some $y, z \in K$ by the connectedness of $K$, $\mu(B_d(x, s_1)) \asymp \mu(B_\theta(x, \theta(x, y)))$ and $\mu(B_d(x, s_2)) \asymp \mu(B_\theta(x, \theta(x, z)))$ by (6.53) and VD of $(K, \theta, \mu)$, and therefore by (6.49), (6.51), the $\eta_{\alpha, \lambda}$-quasisymmetry of $\theta$ to $d$ and (3.5),

$$\frac{\mu(B_d(x, s_2)) \text{Cap}(B_d(x, s_1), K \setminus B_d(x, A_1 s_1))}{\mu(B_d(x, s_1)) \text{Cap}(B_d(x, s_2), K \setminus B_d(x, A_1 s_2))} \preceq \frac{d(x, z)^{d_\ell - d_\mu} \mu(B_\theta(x, \theta(x, z)))}{d(x, y)^{d_\ell - d_\mu} \mu(B_\theta(x, \theta(x, y)))}$$

$$\preceq \left( \frac{\theta(x, z)}{\theta(x, y)} \right)^\beta \in \left[ \frac{1}{2\lambda} \left( \frac{s_2}{s_1} \right)^{1/\alpha}, 2^{2\alpha} \left( \frac{s_2}{s_1} \right)^{\alpha} \right].$$

(6.57)

It follows from (6.57) that [BCM, Proposition 6.17] is applicable to $\mu$ and implies that $\mu$ is admissible with respect to $(K, d, m, \mathcal{E}, \mathcal{F}$). In particular, $(K, \theta, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu)$ satisfies EHI. Finally, to show $\text{cap}(\beta)$ for $(K, \theta, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu)$, let $A_3, A_4 \in (1, \infty)$ satisfy $A_4 \geq 2$, and choose $A_6, A_7, A_8 \in (1, \infty)$ with $A_7 > A_2$ and $A_8 > A_4$ so that (6.54), (6.55) and (6.56) hold. Again thanks to the quasisymmetry of $\theta$ to $d$ and (3.5), by taking $A_8$ large enough we may further assume that $d(x, y) < \text{diam}(K, d)/A_7$ for any $x, y \in K$ with $\theta(x, y) < \text{diam}(K, \theta)/A_8$. Let $(x, s) \in K \times (0, \text{diam}(K, \theta)/(A_5 A_8))$, so that $\theta(x, y) = s = \theta(x, z)/A_5$ for some $y, z \in K$ by the connectedness of $K$. Then by (6.54), (6.55), (6.51) and (6.49),

$$\text{Cap}(B_\theta(x, s), K \setminus B_\theta(x, A_3 s)) \geq \text{Cap}(B_d(x, d(x, y)/A_6), K \setminus B_d(x, A_6 d(x, y)))$$

$$\asymp d(x, y)^{d_\ell - d_w} \asymp \frac{\mu(B_\theta(x, \theta(x, y)))}{\theta(x, y)^\beta} = \frac{\mu(B_\theta(x, s))}{s^\beta},$$

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and similarly by (6.56), (6.53), (6.51), (6.49) and VD of (K, θ, μ),

\[
\text{Cap}(B_θ(x, s), K \setminus B_θ(x, A_3s)) \approx \text{Cap}(B_θ(x, θ(x, z)/A_3), K \setminus B_θ(x, A_3θ(x, z)))
\]
\[
\leq \text{Cap}(B_d(x, d(x, z)), K \setminus B_d(x, A_1d(x, z)))
\]
\[
\approx d(x, z)^{d_1-d_w} \leq \frac{\mu(B_θ(x, θ(x, z)))}{\theta(x, z)^β} \approx \frac{\mu(B_θ(x, s))}{s^β},
\]
proving (6.52) for \((x, s) \in K \times (0, \text{diam}(K, θ)/(A_2A_8))\), namely \(\text{cap}(β)\) for \((K, θ, μ, \mathcal{E}_μ, \mathcal{F}_μ)\).

Now \((K, θ, μ, \mathcal{E}_μ, \mathcal{F}_μ)\) satisfies \(\Phi(β)\) by Theorem 4.4, showing Theorem 6.43-(a).

**Proof.** [Proof of Theorem 6.43] By Proposition 6.44, it suffices to prove the equivalence of Proposition 6.44-(c) and (b). We can verify it in exactly the same way as the proof of Proposition 6.10, by considering the scale \(S = \{Λ_s\}_{s \in (0,1]}\) on Σ defined by \(Λ_1 := \{\emptyset\}\) and

\[
Λ_s := \{w \mid w = w_1 \ldots w_n \in W_* \setminus \{\emptyset\}, \ l^{-|w|} > s \geq l^{-|w|}\}
\]

(6.58)

for each \(s \in (0,1)\), which clearly satisfies (6.20), and by using instead of \(R_ε\) the Euclidean metric \(d\) on \(K\), which is easily seen to satisfy (6.17) with \(d\) in place of \(R_ε\); note that since (6.18) needs to be replaced by (6.49) we also need to replace \(R_ε(x, y)µ(B_{R_ε}(x, R_ε(x, y)))\) in (6.24) by \(d(x, y)^{d_w-d_1}µ(B_d(x, d(x, y)))\).

□

By virtue of Theorem 6.43, the whole of Subsection 6.2 can be easily adapted for the present case with just slight modifications in the proofs, and below we explicitly state the details of the adaptation for concreteness.

**Definition 6.45.** For each \(η \in \text{Homeo}^+\) and \(C \in (1,∞)\) we define \(\mathcal{M}_2(η, C) = \mathcal{M}_2(N, l, S; η, C) \subset C(K \times K) \times \mathcal{P}(K)\) by

\[
\mathcal{M}_2(η, C) := \left\{ (θ, μ) \mid \text{θ is a metric on K η-quasisymmetric to } d, \ μ \in \mathcal{P}(K), \ C^{-1} \leq r_wμ(K_w)/((\text{diam}(K_w, θ))^2 \leq C \text{ for any } w \in W_* \right\},
\]

(6.59)

and we also set \(\mathcal{M}_2 := \mathcal{M}_2(N, l, S) := \bigcup_{η \in \text{Homeo}^+, \ C \in (1,∞)} \mathcal{M}_2(η, C)\).

Since \(μ\) is a Radon measure on \(K\) with full support for any \((θ, μ) \in \mathcal{M}_2\), Theorem 6.43 with \(β = 2\) means that

for the MMD space \((K, d, μ, \mathcal{E}, \mathcal{F})\) the value \(d_{cw} = 2\) is attained by \((K, θ, μ, \mathcal{E}_μ, \mathcal{F}_μ)\) for some \(θ \in \mathcal{J}(K, d)\) and some \(μ \in \mathcal{A}(K, d, μ, \mathcal{E})\) if and only if \(\mathcal{M}_2 \neq \emptyset\).

(6.60)

The following is the main result of this subsection. We take arbitrary \(η \in \text{Homeo}^+\) and \(C \in (1,∞)\), define \(\tilde{η} \in \text{Homeo}^+\) by \(\tilde{η}(t) := 1/η^{-1}(t^{-1}) \ (\tilde{η}(0) := 0)\) and fix them throughout the rest of this subsection.

**Theorem 6.46.** If \(\mathcal{M}_2(η, C) \neq \emptyset\), then there exist \(h \in \mathcal{F}\) which is \(\mathcal{E}\)-harmonic on \(K \setminus V_0\) and a metric \(θ_h\) on \(K\) such that \((θ_h, Γ(h, h)) \in \mathcal{M}_2(η, C)\).

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The rest of this subsection is devoted to a brief sketch of the proof of Theorem 6.46, which goes in exactly the same way as that of Theorem 6.16 except for slight modifications explained in some detail below.

**Proposition 6.47.** $M_2(\eta, C)$ is a compact subset of $C(K \times K) \times P(K)$.

**Proof.** The proof of Proposition 6.17 remains valid also in this case, except that $R_E$ needs to be replaced by $d$ and that $s$ in the last paragraph needs to be defined as $s := l^{-|w|}$. □

**Corollary 6.48.** Let \( \{(\theta_n, \mu_n)\}_{n\in\mathbb{N}} \subset M_2(\eta, C) \), $\mu \in P(K)$ and suppose that \( \{\mu_n\}_{n\in\mathbb{N}} \) converges to $\mu$ in $P(K)$. Then there exists a metric $\theta$ on $K$ such that $(\theta, \mu) \in M_2(\eta, C)$ and some subsequence of $\{\theta_n\}_{n\in\mathbb{N}}$ converges to $\theta$ in $C(K \times K)$.

**Proof.** The proof of Corollary 6.18 remains valid on the basis of Proposition 6.47. □

**Lemma 6.49.** Let $(\theta, \mu) \in M_2(\eta, C)$, $w \in W^*$ and define $(\theta_w, \mu_w) \in C(K \times K) \times P(K)$ by \( (6.36) \). Then $(\theta_w, \mu_w) \in M_2(\eta, C)$.

**Proof.** The proof of Lemma 6.19 remains valid also in this case. □

**Lemma 6.50.** Let $w \in W^*$. Then $\int_K |u \circ F_w| \, dm = (\#S)^{|w|} \int_{K_w} |u| \, dm$ for any Borel measurable function $u : K \to [-\infty, \infty]$, and hence a bounded linear operator from $L^2(K, m)$ to itself is defined by $u \mapsto u \circ F_w$. Moreover, $u \circ F_w \in \mathcal{F}$ and (6.7) holds for any $u, v \in \mathcal{F}$.

**Proof.** The former assertions are immediate from $m = (\#S)^{|w|} m \circ F_w$, and the latter follows from (6.6), (6.7) and the regularity of $(\mathcal{E}, \mathcal{F})$; see [Kaj20, Proof of Lemma 3.3]. □

**Lemma 6.51.** Suppose that $M_2 \neq \emptyset$. Let $u \in \mathcal{F}$ and $w \in W^*$. Then $\Gamma(u, u)(F_w(A)) = r_w^{-1} \Gamma(u \circ F_w, u \circ F_w)(A)$ for any Borel subset $A$ of $K$, and in particular $\Gamma(u, u)(K_w) = r_w^{-1} \mathcal{E}(u \circ F_w, u \circ F_w)$. Moreover, if $\Gamma(u, u)(K_w) > 0$, then (6.37) holds for any Borel subset $A$ of $K$.

**Proof.** By (6.6), (6.7) and [Hin05, Lemma 3.11-(ii)], for any $v \in \mathcal{F}$ and any $n \in \mathbb{N} \cup \{0\},$

\[
\Gamma(v, v)(A) = \sum_{\tau \in W_n} \frac{1}{r_\tau} \Gamma(v \circ F_\tau, v \circ F_\tau)(F_\tau^{-1}(A)) \text{ for any Borel subset } A \text{ of } K. 
\]  \( (6.61) \)

By $M_2 \neq \emptyset$ we can take $(\theta, \mu) \in M_2$, then $\mu$ is a minimal energy-dominant measure of $(\mathcal{E}, \mathcal{F})$ by Theorem 6.43 and Proposition 2.11-(b), thus $\Gamma(v, v) \ll \mu$ for any $v \in \mathcal{F}$, and hence

\[
\Gamma(v, v)(V_0) = 0 \quad \text{for any } v \in \mathcal{F} 
\]  \( (6.62) \)
since $\mu(V_0) = 0$ by Theorem 6.43. Recalling that for any $\tau \in W_w \setminus \{w\}$ we have $K_\tau \cap K_w = F_\tau(V_0) \cap F_w(V_0)$ by Proposition 6.36 and [Kig01, Proposition 1.3.5-(2)] and hence $F_\tau^{-1}(K_w) = F_\tau^{-1}(K_\tau \cap K_w) \subset V_0$, we see from (6.61) and (6.62) that for any Borel subset $A$ of $K$,

$$\Gamma(u, u)(F_w(A)) = \sum_{\tau \in W_w} 1_{r_\tau}(u \circ F_\tau, u \circ F_\tau)(F_\tau^{-1}(F_w(A)))) =\frac{1}{r_w}(u \circ F_w, u \circ F_w)(A).$$

The rest of the proof goes in exactly the same way as that of Lemma 6.20. \hfill $\Box$

**Remark 6.52.** In fact, (6.62) holds without supposing $\mathcal{M}_2 \neq \emptyset$ by [Hin13, Proposition 4.15] and hence so does Lemma 6.51. The proof of (6.62) presented in [Hin13, Section 5], however, is long and difficult, and since we later use Lemma 6.51 only under the supposition that $\mathcal{M}_2 \neq \emptyset$, we have decided to suppose it explicitly to keep our present arguments independent of the demanding result [Hin13, Proposition 4.15].

**Lemma 6.53.** Suppose that $\mathcal{M}_2 \neq \emptyset$, let $(\theta, \mu) \in \mathcal{M}_2$, $u \in \mathcal{F}$ and set $f := d\Gamma(u, u)/d\mu$. Then $\mu$-a.e. $x \in K$ is a $(d, \mu)$-Lebesgue point for $f$, i.e., satisfies

$$\lim_{s \downarrow 0} \frac{1}{\mu(B_d(x, s))} \int_{B_d(x, s)} |f(y) - f(x)| d\mu(y) = 0. \quad (6.63)$$

**Proof.** The proof of Lemma 6.21 remains valid also in this case, except that $R_\xi$ needs to be replaced by $d$. \hfill $\Box$

**Lemma 6.54.** Suppose that $\mathcal{M}_2 \neq \emptyset$, let $(\theta, \mu) \in \mathcal{M}_2$, $u \in \mathcal{F}$, let $f : K \to [0, \infty)$ be a Borel measurable $\mu$-version of $d\Gamma(u, u)/d\mu$ and let $x \in K$ satisfy (6.63). Then (6.39) holds for any $\omega \in \pi^{-1}(x)$ and any $w \in W_w$.

**Proof.** The proof of Lemma 6.22 remains valid also in this case, except that $R_\xi$ needs to be replaced by $d$. \hfill $\Box$

**Proposition 6.55.** Suppose that $\mathcal{M}_2 \neq \emptyset$, let $(\theta, \mu) \in \mathcal{M}_2$, $u \in \mathcal{F}$, let $f : K \to [0, \infty)$ be a Borel measurable $\mu$-version of $d\Gamma(u, u)/d\mu$, let $x \in K$ satisfy (6.63) and $f(x) > 0$, and let $\omega \in \pi^{-1}(x)$. For each $n \in \mathbb{N} \cup \{0\}$, define $\mu_n := \mu_{[\omega]_n} \in \mathcal{P}(K)$ by (6.36) with $w = [\omega]_n$ and, noting that $\Gamma(u, u)(K_{[\omega]_n}) > 0$ by Lemma 6.54, define $u_n := u_{[\omega]_n} \in \mathcal{F}$ by (6.37) with $w = [\omega]_n$. If $v \in \mathcal{F}$ and $\{n_k\}_{k \in \mathbb{N}} \subset \mathbb{N}$ is strictly increasing and satisfies $\lim_{k \to \infty} \mathcal{E}(v - u_{n_k}, v - u_{n_k}) = 0$, then $\Gamma(v, v) \in \mathcal{P}(K)$ and $\{\mu_{n_k}\}_{k \in \mathbb{N}}$ converges to $\Gamma(v, v)$ in $\mathcal{P}(K)$.

**Proof.** The proof of Proposition 6.23 remains valid also in this case, except that it is because of $\mathcal{M}_2 \neq \emptyset$, (6.62) and Lemma 6.51 that $\Gamma(v, v)(F_w(V_0)) = 0$ for any $w \in W_w$. \hfill $\Box$

To conclude the proof of Theorem 6.46, we need the following two lemmas.
**Lemma 6.56.** The inclusion map $\mathcal{F} \hookrightarrow L^2(K, m)$ is a compact linear operator from $(\mathcal{F}, \mathcal{E}_1)$ to $L^2(K, m)$, and there exists $C_P \in (0, \infty)$ such that for any $u \in \mathcal{F}$,

$$\int_K |u - \int_K u\, dm|^2 \, dm \leq C_P \mathcal{E}(u, u).$$

(6.64)

**Proof.** The compactness of the inclusion map $\mathcal{F} \hookrightarrow L^2(K, m)$ follows from Theorem 6.42 and [Dav, Corollary 4.2.3 and Exercise 4.2], and the existence of $C_P \in (0, \infty)$ satisfying (6.64) for any $u \in \mathcal{F}$ is implied by [Dav, Theorems 4.5.1 and 4.5.3] and the fact that $\{u \in \mathcal{F} \mid \mathcal{E}(u, u) = 0\} = \mathbb{R} \mathbf{1}_K$ by Theorem 6.42 and [CF, Theorem 2.1.11].

**Lemma 6.57** (Reverse Poincaré inequality). There exists $C_{RP} \in (0, \infty)$ such that for any $(x, s) \in K \times (0, \infty)$, any $a \in \mathbb{R}$ and any function $h \in \mathcal{F} \cap L^\infty(K, m)$ that is $\mathcal{E}$-harmonic on $B_d(x, 2s)$,

$$\int_{B_d(x, s)} d\Gamma(h, h) \leq \frac{C_{RP}}{s^{d_w}} \int_{B_d(x, 2s) \setminus B_d(x, s)} |h - a|^2 \, dm.$$  

(6.65)

**Proof.** This is a special case of [KM, Lemma 3.3] with $\Psi(s) = s^{d_w}$, whose assumption $\text{CS}(\Psi)$ is implied by Theorem 6.42 and [AB, Theorem 5.5].

**Proof.** [Proof of Theorem 6.46] By the assumption $\mathcal{M}_2(\eta, C) \neq \emptyset$ we can take $(\theta, \mu) \in \mathcal{M}_2(\eta, C)$. Choose $u \in \mathcal{F} \cap L^\infty(K, m)$ which is $\mathcal{E}$-harmonic on $K \setminus V_0$ and satisfies $\mathcal{E}(u, u) > 0$; such $u$ exists, e.g., by [Kaj20, Propositions 3.8 and 3.11]. Let $f : K \to [0, \infty)$ be a Borel measurable $\mu$-version of $d\Gamma(u, u)/d\mu$. Then $\mu(f^{-1}((0, \infty))) > 0$ by $\int_K f \, d\mu = \Gamma(u, u)(K) = \mathcal{E}(u, u) > 0$, and therefore by Lemma 6.53 there exists $x \in K$ with the properties (6.63) and $f(x) > 0$. Let $\omega \in \pi^{-1}(x)$, and for each $n \in \mathbb{N} \cup \{0\}$, as in Proposition 6.55 define $(\theta_n, \mu_n) := (\theta_{|\omega|n}, \mu_{|\omega|n}) \in C(K \times K) \times \mathcal{P}(K)$ by (6.36) and $u_n := u_{|\omega|n} \in \mathcal{F} \cap L^\infty(K, m)$ by (6.37) with $w = |\omega|n$, so that $\{(\theta_n, \mu_n)\}_{n \in \mathbb{N} \cup \{0\}} \subset \mathcal{M}_2(\eta, C)$ by Lemma 6.49 and, for any $n \in \mathbb{N} \cup \{0\}$, $u_n$ is $\mathcal{E}$-harmonic on $K \setminus V_0$ by (6.6) and (6.7) and satisfies $\mathcal{E}(u_n, u_n) = 1$. Then setting $v_n := u_n - \int_K u_n \, dm \in \mathcal{F}$ for each $n \in \mathbb{N} \cup \{0\}$, by $\mathcal{E}(1_K, 1_K) = 0$ and Lemma 6.56 we have $\mathcal{E}(v_n, v_n) \leq C_p + 1$ for any $n \in \mathbb{N} \cup \{0\}$. Hence by [Yos, Chapter V, Section 2, Theorem 1] there exist $h \in \mathcal{F}$ and a strictly increasing sequence $\{n_j\}_{j \in \mathbb{N}} \subset \mathbb{N}$ such that $\{v_{n_j}\}_{j \in \mathbb{N}}$ converges weakly to $h$ in the Hilbert space $(\mathcal{F}, \mathcal{E}_1)$, which in combination with the compactness of the inclusion map from $(\mathcal{F}, \mathcal{E}_1)$ to $L^2(K, m)$ stated in Lemma 6.56 implies further that

$$\lim_{j \to \infty} \int_K |h - v_{n_j}|^2 \, dm = 0.$$  

(6.66)

Moreover, by $\{(\theta_{n_j}, \mu_{n_j})\}_{j \in \mathbb{N}} \subset \mathcal{M}_2(\eta, C)$, Proposition 6.47 and the metrizability of $C(K \times K) \times \mathcal{P}(K)$ we can choose a strictly increasing sequence $\{j_k\}_{k \in \mathbb{N}} \subset \mathbb{N}$ such that $\{(\theta_{n_k}, \mu_{n_k})\}_{k \in \mathbb{N}}$ converges to some $(\vartheta, \nu) \in \mathcal{M}_2(\eta, C)$ in $C(K \times K) \times \mathcal{P}(K)$, where $n_k := n'_{j_k}$. Now we claim that

$$\lim_{k \to \infty} \mathcal{E}(h - u_{n_k}, h - u_{n_k}) = 0.$$  

(6.67)
If (6.67) holds, then $h$ is $\mathcal{E}$-harmonic on $K \setminus V_0$ since $u_{nk}$ is so for any $k \in \mathbb{N}$, Proposition 6.55 implies that $\{\mu_{nk}\}_{k \in \mathbb{N}}$ converges to $\Gamma(h, h) \in \mathcal{P}(K)$ in $\mathcal{P}(K)$, hence $\Gamma(h, h) = \nu$ and thus $(\vartheta, \Gamma(h, h)) = (\vartheta, \nu) \in \mathcal{M}_2(\eta, C)$, which proves the desired assertion.

Thus it remains to show (6.67). To this end, recalling that $K \setminus V_0$ is open in $K$ and non-empty, let $\tau \in W_\ast$, $z_\tau \in K_\tau$ and $s_\tau \in (0, 1)$ satisfy $K_\tau \subset B_d(z_\tau, s_\tau) \subset B_d(z_\tau, 2s_\tau) \subset K \setminus V_0$. Then for any $v \in \mathcal{F} \cap L^\infty(K, m)$ that is $\mathcal{E}$-harmonic on $K \setminus V_0$, by Lemmas 6.51 and 6.57 we have

$$r_\tau^{-1}\mathcal{E}(v \circ F_\tau, v \circ F_\tau) = \Gamma(v, v)(K_\tau) \leq \Gamma(v, v)(B_d(z_\tau, s_\tau)) \leq \frac{C_{RP}}{s_\tau^d} \int_K \left| v - \int_K v \, dm \right|^2 \, dm,$$

(6.68)

which with $v = v_{nj} - v_{nk}$ for $j, k \in \mathbb{N}$, together with (6.66) and Lemma 6.50, shows that $\{v_{nk} \circ F_\tau\}_{k \in \mathbb{N}}$ is a Cauchy sequence in $(\mathcal{F}, \mathcal{E}_1)$ converging in $L^2(K, m)$ to $h \circ F_\tau$, whence

$$\Gamma(h - u_{nk}, h - u_{nk})(K_\tau) = r_\tau^{-1}\mathcal{E}(h \circ F_\tau - v_{nj} \circ F_\tau, h \circ F_\tau - v_{nk} \circ F_\tau) \xrightarrow{k \to \infty} 0$$

(6.69)

with the equality implied by Lemma 6.51 and $\mathcal{E}(1_K, 1_K) = 0$. On the other hand, for any $w \in W_\ast$, combining (6.37) from Lemma 6.51, (6.39) from Lemma 6.54, $f(x) > 0$ and the convergence of $\{\mu_{nk}\}_{k \in \mathbb{N}}$ to $\nu$ in $\mathcal{P}(K)$, we obtain

$$\limsup_{k \to \infty} \frac{\Gamma(u, u)(K_{[w]nk_w})}{\mu(K_{[w]nk_w})} = \limsup_{k \to \infty} \frac{\Gamma(u, u)(K_{[w]nj})}{\mu(K_{[w]nj})} \leq \nu(K_w),$$

(6.70)

Finally, let $n \in \mathbb{N}$, set $V_{0,n} := \bigcup_{w \in W_n, K_w \cap V_0 \neq \emptyset} K_w$ and $W_1^n := \{\tau \in W_{n+1} \mid K_\tau \not\subset V_{0,n}\}$, so that $K = V_{0,n} \cup \bigcup_{\tau \in W_1^n} K_\tau$ and each $\tau \in W_1^n$ satisfies (6.69) since $K_\tau \subset B_d(z, l^{-n-1}) \subset B_d(z, 2l^{-n-1}) \subset K \setminus V_0$ for any $z \in K_\tau$. Then recalling that $\nu(F_\tau(V_0)) = 0$ for any $w \in W_\ast$ by $(\vartheta, \nu) \in \mathcal{M}_2(\eta, C)$ and Theorem 6.43 and that $\Gamma(v, v)(F_w(V_0)) = 0$ for any $v \in \mathcal{F}$ and any $w \in W_\ast$, by $\mathcal{M}_2(\eta, C) \neq \emptyset$, (6.62) and Lemma 6.51, we see from $K = V_{0,n} \cup \bigcup_{\tau \in W_1^n} K_\tau$, [Kig01, Proposition 1.3.5-(2)], (6.69) for $\tau \in W_1^n$ and (6.70) that

$$\limsup_{k \to \infty} \mathcal{E}(h - u_{nk}, h - u_{nk})(K) = \limsup_{k \to \infty} \Gamma(h - u_{nk}, h - u_{nk})(K)$$

$$\leq \limsup_{k \to \infty} \left( \Gamma(h - u_{nk}, h - u_{nk})(V_{0,n}) + \sum_{\tau \in W_1^n} \Gamma(h - u_{nk}, h - u_{nk})(K_\tau) \right)$$

$$\leq \limsup_{k \to \infty} \left( 2\Gamma(h, h)(V_{0,n}) + 2\Gamma(u_{nk}, u_{nk})(V_{0,n}) \right)$$

$$\leq 2\Gamma(h, h)(V_{0,n}) + 2\sum_{w \in W_n, K_w \cap V_0 \neq \emptyset} \Gamma(u_{nk}, u_{nk})(K_w)$$

$$\leq 2\Gamma(h, h)(V_{0,n}) + 2\sum_{w \in W_n, K_w \cap V_0 \neq \emptyset} \nu(K_w)$$

$$= 2\Gamma(h, h)(V_{0,n}) + 2\nu(V_{0,n}) \xrightarrow{n \to \infty} 2\Gamma(h, h)(V_0) + 2\nu(V_0) = 0,$$
which shows (6.67) and thereby completes the proof of Theorem 6.46.

\[ \square \]

**Remark 6.58.** Note that the relatively short proof of (6.67) above is enabled by the supposition of the existence of \((\theta, \mu) \in \mathcal{M}_2\), which implies (6.62), and the property (6.63), which is guaranteed by VD of \((K, d, \mu)\) and yields (6.39) of Lemma 6.54 and thus (6.70). In fact, Hino [Hin13, Proposition 4.18] has obtained a result similar to (6.67) without assuming \(\mathcal{M}_2 \neq \emptyset\), at the price of its long difficult proof presented in [Hin13, Section 5].

## 7 Further remarks and open questions

We collect some open questions that are related to our work.

**Problem 7.1.** Does \(d_{cw} < \infty\) characterize the elliptic Harnack inequality for symmetric jump process?

It is not clear if the equivalence between (a) and (b) in Theorem 2.10 extends to jump processes. Despite the progress made in the diffusion case, the characterization and stability of the elliptic Harnack inequality is still open for jump processes.

**Problem 7.2.** Is the conformal walk dimension attained for the Brownian motion on the standard two-dimensional Sierpiński carpet?

By Theorem 6.46, if the conformal walk dimension is attained then there exists a non-constant function \(h\) harmonic on the complement of \(V_0 = \partial [0, 1]^2\) such that the energy measure \(\Gamma(h, h)\) satisfies the volume doubling property with respect to the Euclidean metric. This motivates the following question as a first step towards Problem 7.2.

**Problem 7.3.** Is there a non-constant function \(h\) harmonic on the complement of \(V_0 = \partial [0, 1]^2\) on the standard two-dimensional Sierpiński carpet such that its energy measure \(\Gamma(h, h)\) satisfies the volume doubling property with respect to the Euclidean metric?

Problem 7.3 appears very challenging since we do not know the answer to the following much simpler question.

**Problem 7.4.** Is there a function \(h\) harmonic on the complement of \(V_0 = \partial [0, 1]^2\) on the standard two-dimensional Sierpiński carpet such that its energy measure \(\Gamma(h, h)\) has full support?

It is tempting to conjecture that the energy measure of every non-constant function harmonic on the complement of \(V_0 = \partial [0, 1]^2\) on the standard two-dimensional Sierpiński carpet has full support. This can be viewed as a unique continuation principle for harmonic functions on the Sierpiński carpet. These questions are open also for any generalized Sierpiński carpet. We also mention this question in a classical setting.

**Problem 7.5.** Characterize all the Gaussian admissible measures for the standard Brownian motion on \(\mathbb{R}^n, n \geq 2\) (see (5.3) for definition).
We expect that there is a version of Theorem 6.16 and 6.46 for Ahlfors regular conformal dimension on self similar spaces. In particular, we expect that if the Ahlfors regular conformal dimension $p > 1$ is attained on a self similar space then it there exists a ‘$p$-harmonic function’ such that its ‘energy measure’ is a $p$-Ahlfors regular measure with respect to a metric in the conformal gauge. This motivates the following problem.

**Problem 7.6.** Construct non-linear analogues of Dirichlet spaces and energy measure on self similar spaces (for example, the Sierpinski carpet).

**References**


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