Conformal Assouad dimension as the critical exponent for combinatorial modulus

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Abstract

The conformal Assouad dimension is the infimum of all possible values of Assouad dimension after a quasisymmetric change of metric. We show that the conformal Assouad dimension equals a critical exponent associated to the combinatorial modulus for any compact doubling metric space. This generalizes a similar result obtained by Carrasco Piaggio for the Ahlfors regular conformal dimension to a larger family of spaces. We also show that the value of conformal Assouad dimension is unaffected if we replace quasisymmetry with power quasisymmetry in its definition.

1 Introduction

The *Assouad dimension* of a metric space (X, d) is defined as

$$\dim_{\mathrm{A}}(X,d) = \inf\left\{ \beta > 0 \;\middle|\; \begin{array}{l} \text{there exists } C>0 \text{ such that } N_r(B(x,R)) \le C\left(\frac{R}{r}\right)^{\beta} \\ \text{for any } x \in X, 0<r<R \end{array} \right\},$$

where $N_r(A)$ denotes the minimum number of balls of radii r required to cover $A \subset X$. Equivalently, Assouad dimension is the infimum of all numbers $\beta > 0$ such that there exists $C > 0$ so that every ball of radius r has at most $C\varepsilon^{-\beta}$ distinct points whose mutual distance is at least εr [Hei, Exercise 10.17]. We refer to the recent book by Fraser [Fra] for a comprehensive background.

We recall the definition of the conformal gauge. This terminology is motivated from the understanding that quasisymmetric maps are an analogue of conformal maps in the context of metric spaces.

Definition 1.1 (Conformal gauge)**.** Let (X, d) be a metric space and θ be another metric on X. We say that d is *quasisymmetric* to θ, if there exists a homeomorphism $\eta : [0,\infty) \to$

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$[0,\infty)$ such that

$$\frac{\theta(x,a)}{\theta(x,b)} \leq \eta\left(\frac{d(x,a)}{d(x,b)}\right) \quad \text{for all triples of points } x,a,b \in X,\ x \neq b.$$

We say that d is *power quasisymmetric* to θ if the homeomorphism η above can be chosen so that $\eta(t) = C(t^\alpha \vee t^{1/\alpha})$ for all $t > 0$, where $C, \alpha \in [1,\infty)$. The *conformal gauge* of a metric space (X,d) is defined as

$$\mathcal{J}(X,d) := \{\theta : X \times X \to [0,\infty) \mid \theta \text{ is a metric on } X,\ d \text{ is quasisymmetric to } \theta\}. \tag{1.1}$$

We define the *power quasisymmetric conformal gauge* of (X,d) as

$$\mathcal{J}_p(X,d) := \{\theta \in \mathcal{J}(X,d) \mid \theta \text{ is power quasisymmetric to } d\}. \tag{1.2}$$

The *conformal Assouad dimension* of (X,d) is defined as

$$\dim_{\mathrm{CA}}(X,d) = \inf\{\dim_{\mathrm{A}}(X,\theta) : \theta \in \mathcal{J}(X,d)\}, \tag{1.3}$$

where $\dim_{\mathrm{A}}(X,\theta)$ denotes the Assouad dimension of (X,θ).

As our main result relates conformal Assouad dimension with combinatorial modulus, we recall the notion of combinatorial modulus and a critical exponent associated to it. The combinatorial p-modulus of a family of curves Γ in a graph $G = (V,E)$ is defined as

$$\mathrm{Mod}_p(\Gamma, G) = \inf\left\{\sum_{v\in V} \rho(v)^p \mid \rho : V \to [0,\infty), \sum_{v\in\gamma} \rho(v) \geq 1 \text{ for all } \gamma \in \Gamma\right\}.$$

Fix parameters $a, \lambda, L > 1$. We choose a sequence $X_k, k \geq 0$ such that X_k is a maximal a^{-k}-separated subset of (X,d) and $X_k \subset X_{k+1}$ for all $k \geq 0$. For each k, we define a graph G_k whose vertex set is X_k and there is an edge between two distinct vertices $x, y \in X_k$ if and only if $B(x, \lambda a^{-k}) \cap B(y, \lambda a^{-k}) \neq \emptyset$. We think of G_k as a sequence of combinatorial approximations of (X,d) at scale a^{-k}. We define

$$M_{p,k}(L) = \sup\{\mathrm{Mod}_p(\Gamma_{k,L}(x), G_{k+n}) \mid x \in X_n, n \geq 0\} \text{ and } M_p(L) = \liminf_{k\to\infty} M_{p,k}(L),$$

where $\Gamma_{k,L}(x)$ is the family of paths in G_{n+k} from $B(x,a^{-n})$ to $B(x, La^{-n})^c$ (see §4 for a detailed definition). The critical exponent corresponding to combinatorial modulus is defined as

$$\mathrm{CE}(X,d) = \inf\{p > 0 \mid M_p(L) = 0\}.$$

It is not difficult to show that $\mathrm{CE}(X,d)$ is well-defined in the sense that $\mathrm{CE}(X,d)$ does not depend on the precise choices of $a, L, \lambda \in (1,\infty)$ and also on the choices of X_k (see Proposition 4.3). Since it only depends on the metric space (X,d), our notation $\mathrm{CE}(X,d)$ is justified.

Our main result is the following theorem.

Theorem 1.2. *Let (X,d) be a compact metric space such that $\dim_{\mathrm{A}}(X,d) < \infty$. Then*

$$\dim_{\mathrm{CA}}(X,d) = \mathrm{CE}(X,d) = \inf\{\dim_{\mathrm{A}}(X,\theta) : \theta \in \mathcal{J}_p(X,d)\}.$$

A similar result was obtained by Carrasco [Car, Theorem 1.3] for the Ahlfors regular conformal dimension and independently in an unpublished work of Keith and Kleiner. These works rely crucially on ideas of Keith and Laakso who first related conformal Assouad dimension to combinatorial modulus [KL]. To state Carrasco's result, we recall the definition of Ahlfors regular conformal dimension and related notions. A Borel measure μ on (X,d) is said to be *p-Ahlfors regular* if there exists $C \geq 1$ such that

$$C^{-1}r^p \leq \mu(B(x,r)) \leq Cr^p \quad \text{for all } x \in X, 0 < r \leq \operatorname{diam}(X,d).$$

Note that if such a p-Ahlfors regular μ exists, then the p-Hausdorff measure is also p-Ahlfors regular. The Ahlfors regular conformal dimension is defined as

$$\dim_{\mathrm{ARC}}(X,d) = \inf\{p > 0 : \theta \in \mathcal{J}(X,d) \text{ and } \mu \text{ is a } p\text{-Ahlfors regular measure on } (X,\theta)\}. \tag{1.4}$$

In (1.4) and (1.3) we adopt the convention that $\inf \emptyset = \infty$. Ahlfors regular conformal dimension is a well-studied notion in complex dynamics and hyperbolic groups; see for example [BK05, BM, CM, HP09, PT, Par]. These notions of conformal dimensions are variants of the one introduced by Pansu [Pan89] and we refer the reader to [MT10] for more background and applications.

To compare our results with earlier ones, we recall the notion of doubling and uniformly perfect metric spaces. A metric space is said to be *doubling*, if there exists $N \in \mathbb{N}$ such that every ball of radius r can be covered by at most N balls of radii $r/2$. It is easy to see that the $\dim_{\mathrm{A}}(X,d) < \infty$ if and only if (X,d) is doubling. A metric space (X,d) is said to be *uniformly perfect* if there exists $C > 1$ such that whenever $B(x,r) \neq X$, we have $B(x,r) \setminus B(x,r/C) \neq \emptyset$. Carrasco's theorem [Car, Theorem 1.3] states that for any compact, doubling, uniformly perfect metric space, the Ahlfors regular conformal dimension is given by

$$\dim_{\mathrm{ARC}}(X,d) = \mathrm{CE}(X,d).$$

The following lemma characterizes the class of metric spaces for which $\dim_{\mathrm{CA}}(X,d)$ and $\dim_{\mathrm{ARC}}(X,d)$ are finite.

Lemma 1.3. *Let (X,d) be a compact metric space. Then*

(a) $\dim_{\mathrm{CA}}(X,d)$ *is finite if and only if* (X,d) *is doubling.*

(b) $\dim_{\mathrm{ARC}}(X,d)$ *is finite if and only if* (X,d) *is doubling and uniformly perfect. Moreover, If* (X,d) *is doubling and uniformly perfect, then* $\dim_{\mathrm{ARC}}(X,d) = \dim_{\mathrm{CA}}(X,d)$ *[MT10, Proposition 2.2.6].*

By Lemma 1.3, our result in Theorem 1.2 generalizes Carrasco's theorem [Car, Theorem 1.3] to doubling metric spaces that are not necessarily uniformly perfect. We refer to [Kig20, Sha] for expositions to Carrasco's work.

One motivation for this work is that conformal Assouad dimension is better behaved than Ahlfors regular conformal dimension. The above lemma shows that conformal Assouad dimension is a meaningful quasisymmetry invariant for a larger class of metric spaces. If (X,d) is a compact metric space and $Y \subset X$, then it is easy to see that

$$\dim_{\mathrm{CA}}(Y,d) \le \dim_{\mathrm{CA}}(X,d).$$

The above inequality is not always true for Ahlfors regular conformal dimension because a subset of uniformly perfect metric space is not necessarily uniformly perfect. Nevertheless, if (X,d) is a compact, doubling, uniformly perfect metric space and $Y \subset X$ is also compact, doubling and uniformly perfect, then

$$\dim_{\mathrm{ARC}}(Y,d) \le \dim_{\mathrm{ARC}}(X,d). \tag{1.5}$$

One way to show (1.5) is to use Lemma 1.3(b) and the analogous inequality for conformal Assouad dimension. Another more involved approach would be to use [Car, Theorem 1.3] and careful choices of hyperbolic fillings for X and Y. The direct approach of restricing an Ahlfors regular metric in $\mathcal{J}(X,d)$ to Y does not work because the restriction of an Ahlfors regular metric on a subset need not be Ahlfors regular. To summarize, conformal Assouad dimension is better behaved because Ahlfors regularity and uniform perfectness do not pass to a subspace.

We briefly discuss the result $\dim_{\mathrm{CA}}(X,d) = \inf\{\dim_{\mathrm{A}}(X,\theta) : \theta \in \mathcal{J}_p(X,d)\}$. Since $\mathcal{J}_p(X,d) \subset \mathcal{J}(X,d)$, the upper bound on $\dim_{\mathrm{CA}}(X,d)$ is obvious but the other inequality is non-trivial as it is possible that $\mathcal{J}_p(X,d) \neq \mathcal{J}(X,d)$ as we recall below. Trotsenko and Väisälä characterize metric spaces for which $\mathcal{J}_p(X,d) = \mathcal{J}(X,d)$. To state their characterization, we recall the notion of *weakly uniformly perfect* spaces. We say that a metric space is *weakly uniformly perfect* if there exists $C > 1$ such that if $B(x,r) \neq X$ for some $x \in X, r > 0$, then either $B(x,r) = \{x\}$ or $B(x,r) \setminus B(x,r/C) \neq \emptyset$. The Trotsenko-Väisälä theorem states that a compact metric space (X,d) satisfies $\mathcal{J}_p(X,d) = \mathcal{J}(X,d)$ if and only if (X,d) is weakly uniformly perfect [TV, Theorems 4.11 and 6.20].

1.1 Outline of the work

To show the estimate $\dim_{\mathrm{CA}}(X,d) \le \mathrm{CE}(X,d)$, we construct a graph which is Gromov hyperbolic called the hyperbolic filling (see §2.3). A theorem of Bonk and Schramm implies that a quasi-isometric change of metric on the hyperbolic filling induces a power quasisymmetric change of metric on its boundary. Roughly speaking, a quasi-isometric change of metric is done using the optimal functions for the combinatorial modulus. This is done in [Car, Theorems 1.1 and 1.2] where the author introduces hypotheses on weight functions on the graph that defines a bi-Lipschitz change of metric in the hyperbolic filling. However the hypotheses introduced in [Car, Theorem 1.1] implies that (X,d) is uniformly perfect as pointed in [Sha, Lemma 6.2]. Since the metric spaces we consider are not necessarily uniformly perfect, we need modify one of the hypothesis so that it is more suitable for bounding the conformal Assouad dimension (see hypothesis (H4) in Theorem 3.3). The key new tool is a modification of a lemma of Vol'berg and Konyagin to construct a p-homogeneous measure on (X, θ), where θ is power quasisymmetric to d and $p > \mathrm{CE}(X, d)$ (see Lemma 3.11 and Proposition 3.16). This along with Theorem 3.1 implies the bound $\dim_{\mathrm{CA}}(X, d) \le \mathrm{CE}(X, d)$. Another key distinction from [Car] is that the metric space is not necessarily uniformly perfect. Therefore by the Trotsenko-Väisälä theorem, this approach need not construct all possible metrics in $\mathcal{J}(X, d)$. Nevertheless, this approach provides the sharp upper bound and also leads to $\dim_{\mathrm{CA}}(X, d) = \inf\{\dim_{\mathrm{A}}(X, \theta) : \theta \in \mathcal{J}_p(X, d)\}$.

For the other bound $\mathrm{CE}(X, d) \le \dim_{\mathrm{CA}}(X, d)$, we use a p-homogeneous measure μ in (X, θ) and $\theta \in \mathcal{J}(X, d)$ for $p > \dim_{\mathrm{CA}}(X, d)$ and define an function ρ for the combinatorial modulus that is similar to [Car, (3.7)]. However some modifications are needed because [Car] uses the uniform perfectness in an essential way to control ρ. Some of the parameters and constants in [Car] depend on the constant associated with the uniform perfectness property. Much of the work is about removing such dependence on uniform perfectness.

2 Hyperbolic filling of a compact metric space

2.1 Gromov hyperbolic spaces and its boundary

Let (Z, d) be a metric space. We recall some basic notions regarding Gromov hyperbolic spaces and refer the reader to [BH, CDP, GH90, Gro87, Vä05] for a detailed exposition. Given three points $x, y, w \in Z$, we define the *Gromov product* of x and y with respect to the base point w as

$$(x|y)_w = \frac{1}{2}(d(x, w) + d(y, w) - d(x, y)).$$

By the triangle inequality, Gromov product is always non-negative. We say that a metric space (Z, d) is *δ-hyperbolic*, if for any four points $x, y, z, w \in Z$, we have

$$(x|z)_w \ge (x|y)_w \wedge (y|z)_w - \delta.$$

We say that (Z, d) is hyperbolic (or d is a hyperbolic metric), if (Z, d) is hyperbolic for some $\delta \in [0, \infty)$. If the above condition is satisfied for a fixed base point $w \in Z$, and arbitrary $x, y, z \in Z$, then (Z, d) is 2δ-hyperbolic [CDP, Proposition 1.2].

We recall the definition of the boundary of a hyperbolic space. Let (Z, d) be a hyperbolic space and let $w \in Z$. A sequence of points $(x_i)_{i \in \mathbb{N}} \subset Z$ is said to *converge at infinity*, if

$$\lim_{i,j \to \infty} (x_i|x_j)_w = \infty.$$

The above notion of convergence at infinity does not depend on the choice of the base point $w \in Z$, because by the triangle inequality $|(x|y)_w - (x|y)_{w'}| \le d(w, w')$.

Two sequences $(x_i)_{i \in \mathbb{N}}, (y_i)_{i \in \mathbb{N}}$ that converge at infinity are said to be *equivalent*, if

$$\lim_{i \to \infty} (x_i|y_i)_w = \infty.$$

This defines an equivalence relation among all sequences that converge at infinity [CDP, §1, Chapter 2]. As before, is easy to check that the notion of equivalent sequences does not depend on the choice of the base point w. The *boundary* ∂Z of (Z,d) is defined as the set of equivalence classes of sequences converging at infinity under the above equivalence relation. If there are multiple hyperbolic metrics on the same set Z, to avoid confusion, we denote the boundary of (Z,d) by $\partial(Z,d)$. The notion of Gromov product can be defined on the boundary as follows: for all $a,b \in \partial Z$

$$(a|b)_w = \sup\left\{\liminf_{i\to\infty}(x_i|y_i)_w : (x_i)_{i\in\mathbb{N}} \in a, (y_i)_{i\in\mathbb{N}} \in b\right\}.$$

By [GH90, Remarque 8, Chapitre 7], if $(x_i)_{i\in\mathbb{N}} \in a, (y_i)_{i\in\mathbb{N}} \in b$, we have

$$(a|b)_w - 2\delta \le \liminf_{i\to\infty}(x_i|y_i)_w \le (a|b)_w.$$

The boundary ∂Z of the hyperbolic space (Z,d) carries a family of metrics. A metric $\rho : \partial Z \times \partial Z \to [0,\infty)$ on ∂Z is said to be a *visual metric* with base point $w \in Z$ and *visual parameter* $\alpha \in (1,\infty)$ if there exists $k_1, k_2 > 0$ such that

$$k_1\alpha^{-(a|b)_w} \le \rho(a,b) \le k_2\alpha^{-(a|b)_w}$$

If a visual metric with base point w and visual parameter α exists, then it can be chosen to be

$$\rho_{\alpha,w}(a,b) := \inf \sum_{i=1}^{n-1} \alpha^{-(a_i|a_{i+1})_w},$$

where the infimum is over all finite sequences $(a_i)_{i=1}^n \subset \partial Z, n \ge 2$ such that $a_1 = a, a_n = b$. Any other visual metric with the same basepoint and visual parameter is bi-Lipschitz equivalent to $\rho_{\alpha,w}$.

Visual metrics exist on hyperbolic metric spaces as we recall now. For any δ-hyperbolic space (Z,d), there exists $\alpha_0 > 1$ (α_0 depends only on δ) such that if $\alpha \in (1,\alpha_0)$, then there exists a visual metric with parameter α [GH90, Chapitre 7], [BoSc, Lemma 6.1]. It is well-known that quasi-isometry between almost geodesic hyperbolic spaces induces a quasisymmetry on their boundaries (the notion of almost geodesic space is given in Definition 2.1). Since this plays a central role in our construction of metric, we recall the relevant definitions and results below.

We say that a map (not necessarily continuous) $f : (X_1,d_1) \to (X_2,d_2)$ between two metric spaces is a *quasi-isometry* if there exists constants $A, B > 0$ such that

$$A^{-1}d_1(x,y) - A \le d_2(f(x),f(y)) \le Ad_1(x,y) + A$$

for all $x, y \in X_1$, and

$$\sup_{x_2\in X_2} d(x_2, f(X_1)) = \sup_{x_2\in X_2}\inf_{x_1\in X_1} d(x_2,f(x_1)) \le B.$$

Definition 2.1. A metric space (Z,d) is k-*almost geodesic*, if for every $x,y \in Z$ and every $t \in [0,d(x,y)]$, there is some $z \in Z$ with $|d(x,z) - t| \le k$ and $|d(y,z) - (d(x,y) - t)| \le k$. We say that a metric space is *almost geodesic* if it is k-almost geodesic for some $k \ge 0$.

Quasi-isometry between hyperbolic spaces induce quasisymmetries on their corresponding boundaries. We recall a result due to Bonk and Schramm below.

Proposition 2.2 ([BoSc, Theorem 6.5 and Proposition 6.3])**.** *Let (Z_1, d_1) and (Z_2, d_2) be two almost geodesic, δ-hyperbolic metric spaces. Let $f : (Z_1, d_1) \to (Z_2, d_2)$ be a quasi-isometry.*

(a) If $(x_i)_{i\in\mathbb{N}} \subset Z_1$ converges at infinity, then $(f(x_i))_{i\in\mathbb{N}} \subset Y$ converges at infinity. If $(x_i)_{i\in\mathbb{N}}$ and $(y_i)_{i\in\mathbb{N}}$ are equivalent sequences in X converging at infinity, then $(f(x_i))_{i\in\mathbb{N}}$ and $(f(y_i))_{i\in\mathbb{N}}$ are also equivalent.

(b) The map $\partial f : \partial Z_1 \to \partial Z_2$ given by $\partial f\left((x_i)_{i\in\mathbb{N}}\right) = (f(x_i))_{i\in\mathbb{N}}$ is well-defined, and is a bijection.

(c) Let $p_1 \in Z_1$ be a base point in Z_1, and let $f(p_1)$ be a corresponding base point in Z_2. Let ρ_1, ρ_2 denote visual metrics (with not necessarily the same visual parameter) on $\partial Z_1, \partial Z_2$ with base points $p_1, f(p_1)$ respectively. Then the induced boundary map $\partial f : (\partial Z_1, \rho_1) \to (\partial Z_2, \rho_2)$ is a power quasisymmetry.

2.2 Geodesic hyperbolic spaces

Let (Z, d) by a geodesic δ-hyperbolic metric space. Recall that (Z, d) is geodesic if for any $x, y \in X$, there exists a curve $\gamma : [0, d(x, y)] \to Z$ such that $\gamma(0) = x, \gamma(d(x, y)) = y$ and $d(\gamma(s), \gamma(t)) = |s - t|$ for all $s, t \in [0, d(x, y)]$. Such a curve is called a geodesic between x and y. For $x, y \in Z$, we denote by $[x, y]$ a *geodesic* between x and y. For $x, y, z \in Z$, we denote by $[x, y, z] = [x, y] \cup [y, z] \cup [z, x]$ a *geodesic triangle* in Z. Recall that a *tripod* is a metric tree with three edges arising from a common *central vertex* such that each edge a is isometric to the closed interval $[0, l(a)]$ for some $l(a) \geq 0$ called the length of the edge a. A tripod is determined up to isometry by the length of the three edges. We allow for the degenerate case where the length of some of the edges could be zero.

Given a geodesic triangle $\Delta = [x, y, z]$, there exists a map $f_\Delta : \Delta \to T_\Delta$ from Δ to a tripod T_Δ such that the restriction of f_Δ to each side of the triangle is an isometry [GH90, Proposition 2]. The *inscribed triple* of a geodesic triangle Δ is defined to be the preimages of the 'central vertex' of the tripod T_Δ under the map f_Δ described above.

Unlike a tripod, a geodesic triangle Δ need not have a canonical center. However, it has a reasonable notion of approximate center. For $K \geq 0$, a point $c \in Z$ is a K-*approximate center* of a geodesic triangle $[x, y, z]$ if c is at a distance at most K from each of the three sides, that is, $d(c, [x, y]) \vee d(c, [y, z]) \vee d(c, [z, x]) \leq K$. The following proposition concerns a few properties of approximate center.

Proposition 2.3. *Let (Z, d) be a geodesic, δ-hyperbolic metric space.*

(a) Each point of the inscribed triple of a geodesic triangle is a 4δ-approximate center.

(b) Any two K-approximate centers c and c' of a geodesic triangle $[x, y, z]$ satisfies $d(c, c') \leq 8K$.

(c) *If c is a K-approximate center of a geodesic triangle $[x, y, z]$ then*

$$|d(x, c) - (y|z)_x| \leq 4K.$$

(d) *If $f : (Z_1, d_1) \to (Z_2, d_2)$ is a quasi-isometry between two geodesic δ-hyperbolic metric spaces and if c is a K_1-approximate center of $[x, y, z]$ then $f(c)$ is a K_2-approximate center of any geodesic triangle $[f(x), f(y), f(z)]$, where K_2 depends only on K_1, δ and the constants associated with the quasi-isometry f. In particular,*

$$\left|d_2(f(x), f(c)) - (f(y)|f(z))_{f(x)}\right| \leq 4K_2.$$

Proof.

(a) By [GH90, Proposition 21, Chapitre 2] each point of the inscribed triple is a 4δ-approximate center.

(b,c) Let c denote a K-approximate center of $[x, y, z]$. Let p_1, p_2, p_3 be the points of the inscribed triple on $[x, y], [y, z], [z, x]$ respectively. Similarly, let q_1, q_2, q_3 be three points on $[x, y], [y, z], [z, x]$ respectively such that $d(c, q_i) \leq A$ for all $i = 1, 2, 3$. This implies that $d(q_i, q_j) \leq 2K$ for all i, j. By the argument in [GH90, Proof of Lemme 20, Chapitre 2] we have

$$d(p_i, q_i) \leq 3K \quad \text{for all } i = 1, 2, 3. \tag{2.1}$$

Since $d(x, p_1) = (y|z)_x$ and $d(p_1, q_1) \leq 3K$, we obtain

$$|d(x, c) - (y|z)_x| = |d(x, c) - d(x, p_1)| \leq d(p_1, q_1) + d(c, q_1) \leq 3K + K = 4K.$$

This concludes the proof of (c). Similarly, $d(c, p_i) \leq d(p_i, q_i) + d(c, q_i) \leq 4K$ for all $i = 1, 2, 3$. Therefore $d(c, c') \leq d(c, p_1) + d(c', p_1) \leq 8K$, and hence (b) holds.

(d) This is an immediate consequence of the geodesic stability under quasi-isometries [GH90, Théorème 11, Chapitre 5] and (c).

□

2.3 Construction of hyperbolic filling

In this section, we recall the construction of a hyperbolic filling of a compact metric space. Let (X, d) be a compact metric space. The construction below is due to A. Bjorn, J. Bjorn and Shanmugalingam [BBS]. Earlier versions of this construction are due to Elek, Bourdon and Pajot [Ele, BP].

Let $\lambda, a \in (1, \infty)$ be two parameters which we call the horizontal and vertical parameter of the hyperbolic filling respectively. We assume that the diameter is normalized so that $\operatorname{diam}(X, d) = \frac{1}{2}$. Let $X_n, n \in \mathbb{N}_{\geq 0}$ be an increasing sequence of maximal a^{-n}-separated subsets of X. In other words, $X_n \subset X_m$ for all $n < m$, any two distinct points in X_n have mutual distance at least a^{-n} and any set strictly larger than X_n has two distinct points whose distance is strictly less than a^{-n}. The vertex set of the graph is $\mathcal{S} = \cup_{n\geq 0}\mathcal{S}_n$, where $\mathcal{S}_n = \{(x,n) : x \in X_n\}$. Two distinct vertices $(x,n),(y,m) \in \mathcal{S}$ are joined by an edge if and only if either $n = m$ and $B(x,\lambda a^{-n}) \cap B(y,\lambda a^{-m}) \neq \emptyset$ or if $|n-m| = 1$ and $B(x,a^{-n})\cap B(y,a^{-m}) \neq \emptyset$. Let D_1 denote the combinatorial (graph) distance on $\mathcal{S}$ defined by the above set of edges. That is $D_1((x,n),(y,m))$ is the minimal number k such that $(x,n) = (x_0,n_0),(x_1,n_1),\ldots,(x_k,n_k) = (y,m)$, where $(x_i,n_i),(x_{i+1},n_{i+1}) \in \mathcal{S}$ is joined by an edge for all $i = 0,1,\ldots,k-1$. It is evident that $(\mathcal{S},D_1)$ is 1-almost geodesic metric space.

We now construct a graph with fewer vertical edges. For each $(x,n) \in \mathcal{S}$, we choose $(y,n-1) \in \mathcal{S}_n$ such that $d(x,y) = \min_{z\in X_{n-1}} d(x,z)$. In this case, we say that $(y,n-1)$ is the *parent* of (x,n) or equivalently, (x,n) is a *child* of $(y,n-1)$. Such a choice of $y \in X_{n-1}$ is not unique but we fix this choice for the remainder of this work. Since X_{n-1} is a maximal $a^{-(n-1)}$ separated subset of X, $d(x,y) \leq a^{-(n-1)}$. Hence $x \in B(x,a^{-n})\cap B(y,a^{-(n-1)}) \neq \emptyset$. In other words every parent and their child is connected by an edge in the graph associated with $(\mathcal{S},D_1)$. We define a new graph whose edges consists of all of the edges between parent and child and those between $(x,n),(y,n) \in \mathcal{S}$ where $B(x,\lambda a^{-n}) \cap B(y,\lambda a^{-n}) \neq \emptyset, x \neq y$. The corresponding graph distance is denoted by D_2. The *set of children of a vertex* v is denoted by

$$C(v) := \{w \in \mathcal{S} : w \text{ is a child of } v\}. \tag{2.2}$$

Note that $C(v) \subset \mathcal{S}_{n+1}$ whenever $v \in \mathcal{S}_n$.

If $(x,n+1)$ and $(y,n+1)$ share an edge in D_2 and if (x_0,n) and (y_0,n) are their respective parents, then $d(x_0,y_0) \leq d(x,y)+d(x,x_0)+d(y,y_0) < 2a^{-n}+2\lambda a^{-n-1}$. Under the assumption $\lambda \geq 2+2\lambda a^{-1}$, we have $D_2((x,n+1),(y,n+1)) \leq 1$, then $D_2((x_0,n),(y_0,n)) \leq 1$ whenever $(x_0,n),(y_0,n)$ are the parents of $(x,n+1),(y,n+1)$ respectively. We say that (x,n) is a *descendant* of (y,k) if $n > k$, and there exists $(z_j,n_j) \in \mathcal{S}$ for $j = 0,\ldots,n-k$ such that $(z_0,n_0) = (y,k),(z_{n-k},n_{n-k}) = (x,n)$, where (z_{i+1},n_{i+1}) is a child of (z_i,n_i) for all $i = 0,\ldots,n-k-1$. For any $n > k \geq 0$ and $v \in \mathcal{S}_k$, the *set of descendants of* v *in generation* n is denoted by

$$\mathcal{D}_n(v) = \{w \in \mathcal{S}_n : w \text{ is a descendant of } v\}. \tag{2.3}$$

The following lemma is an analogue of [Car, Lemma 2.2].

Lemma 2.4. *Let $\lambda, a > 1$ be horizontal and vertical parameters of the hyperbolic filling respectively.*

(a) If $(z,n+1)$ is a child of (x,n), then $d(x,z) < a^{-n}$. If (y,k) is a descendant of (x,n) (for some $k > n$), then
$$d(x,y) < \frac{a}{a-1}a^{-n}.$$

(b) If $\lambda \geq 2+2\lambda a^{-1}$ and $D_2((x,n+1),(y,n+1)) \leq 1$, then $D_2((x_0,n),(y_0,n)) \leq 1$, where $(x_0,n),(y_0,n)$ are the parents of $(x,n+1),(y,n+1)$ respectively. Similarly, if $\lambda \geq 2+4\lambda a^{-1}$ and $D_2((x,n+1),(y,n+1)) \leq 2$, then $D_2((x_0,n),(y_0,n)) \leq 1$, where $(x_0,n),(y_0,n)$ are the parents of $(x,n+1),(y,n+1)$.

(c) *Let $\lambda \geq 6$. If $(x, n+1), (y, n+1) \in \mathcal{S}_{n+1}$ such that $d(x,y) \leq 4a^{-n}$. If $(x_0, n), (y_0, n) \in \mathcal{S}_n$ are the parents of $(x, n+1), (y, n+1)$ respectively, then $D_2((x_0, n), (y_0, n)) \leq 1$.*

(d) *If $\lambda > 1 + a^{-1}$, we have $D_1 \leq D_2 \leq 2D_1$.*

(e) *Let $\lambda > 1 + a^{-1}$. Let $w \in \mathcal{S}_{n+1}$ and $u, v \in \mathcal{S}_n$ be such that $D_1(u, w) = 1$ and $D_2(v, w) = 1$. Then $D_2(u, v) \leq 1$.*

Proof.

(a) Since X_n is maximal a^{-n}-subset of X, every point $z \in X$ satisfies $d(z, X_n) < a^{-n}$. This shows the first claim. If (y, k) is a descendant of (x, n) by the first claim and triangle inequality $d(x, y) \leq \sum_{i=n}^{k} a^{-i} < \frac{a}{a-1} a^{-n}$.

(b) Since $d(x_0, y_0) \leq d(x, y) + d(x, x_0) + d(y, y_0) < 2a^{-n} + 2\lambda a^{-n-1}$, we have $\{x_0, y_0\} \subset B(x_0, \lambda a^{-n}) \cap B(y_0, \lambda a^{-n}) \neq \emptyset$ for any λ such that $\lambda \geq 2 + 2\lambda a^{-1}$. Hence $D_2((x_0, n), (y_0, n)) \leq 1$. The other claim follows from a similar argument.

(c) Since $d(x_0, y_0) \leq d(x, y) + d(x, x_0) + d(y, y_0) < 4a^{-n} + a^{-n} + a^{-n} = 6a^{-n}$, we have $\{x_0, y_0\} \subset B(x_0, \lambda a^{-n}) \cap B(y_0, \lambda a^{-n}) \neq \emptyset$ whenever $\lambda \geq 6$. Therefore $D_2((x_0, n), (y_0, n)) \leq 1$.

(d),(e) Since every edge in the graph corresponding to $(\mathcal{S}, D_2)$ is contained in the graph corresponding to $(\mathcal{S}, D_2)$, we have $D_2 \geq D_1$.

On the other hand, if there is an edge in $(\mathcal{S}, D_1)$[1] which is not present in $(\mathcal{S}, D_2)$, then it must be between some $(x, n), (y, n+1)$ such that $n \in \mathbb{N}_{\geq 0}$ and that the parent of $(y, n+1)$ is (z, n) where $z \neq x$. In this case $d(x, y) \leq a^{-n-1} + a^{-n}$ (since $B(x, a^{-n}) \cap B(y, a^{-n-1}) \neq \emptyset$). Therefore if $\lambda > 1 + a^{-1}$, there would be an edge between (x, n) and (z, n) in both graphs (since $y \in B(x, \lambda a^{-n}) \cap B(z, \lambda a^{-n})$). This implies that

$$D_1 \leq D_2 \leq 2D_1, \quad \text{whenever } \lambda \geq 1 + a^{-1}. \tag{2.4}$$

□

We recall the relevant properties the metric spaces $(\mathcal{S}, D_1)$ and $(\mathcal{S}, D_2)$. By the choice of the diam(X, d), there is an unique point $x_0 \in X_0$. We choose $v_0 := (x_0, 0)$ as the base point of the metric spaces $(\mathcal{S}, D_1)$ and $(\mathcal{S}, D_2)$. We denote the Gromov product with respect to the basepoint v_0 in $(\mathcal{S}, D_1)$ and $(\mathcal{S}, D_2)$ by $(\cdot|\cdot)_1, (\cdot|\cdot)_2$ respectively. The key point in the following result is that the hyperbolicity constant δ depends only on a and λ unlike the analogous result in [BP, Car] where δ also depends on the constant associated with the uniform perfectness property (see [Car, Remark after Proposition 2.1]).

Proposition 2.5. *(Cf. [BBS, Lemma 3.3 and Theorem 3.4]) Let (X, d) be a compact metric space and let a, λ denote the vertical and horizontal parameters respectively of the hyperbolic filling. Then the hyperbolic filling $(\mathcal{S}, D_1)$ satisfies the following properties*

[1] Here we abuse notation and use $(\mathcal{S}, D_i)$ to denote the graph, for $i = 1, 2$.

(a) *For any* $v=(z,n), w=(y,m)\in\mathcal{S}$, *we have*

$$\frac{a-1}{4\lambda a}\left(d(z,y)+a^{-n}+a^{-m}\right)\le a^{-(v|w)_1}\le\frac{a^{5/2}}{\lambda-1}\left(d(z,y)+a^{-n}+a^{-m}\right)$$

In particular, if $a,\lambda\in[2,\infty)$ *and* $a\ge\lambda$, *then*

$$\left|(v|w)_1+\frac{\log(d(x,y)+a^{-m}+a^{-n})}{\log a}\right|\le 4.$$

(b) $(\mathcal{S},D_1)$ *is* δ*-hyperbolic, where* $\delta=2\frac{\log\left(\frac{8\lambda a^{7/2}}{(a-1)(\lambda-1)}\right)}{\log a}$. *In particular, if* $a,\lambda\in[2,\infty)$ *and* $a\ge\lambda$ *implies that* δ *can be chosen to be* 15.

Proof.

(a) The first estimate follows from [BBS, Proof of Lemma 3.3]. The second conclusion is a consequence of the estimate

$$\max\left(\frac{\log(a^{5/2}/(\lambda-1))}{\log a},\frac{\log\left(\frac{4\lambda a}{a-1}\right)}{\log a}\right)\le 4\quad\text{whenever } a\ge\lambda\ge 2.$$

(b) The δ-hyperbolicity follows from the proof of [BBS, Theorem 3.4] along with [CDP, Proposition 1.2]. For the second conclusion, observe that

$$2\frac{\log\left(\frac{8\lambda a^{7/2}}{(a-1)(\lambda-1)}\right)}{\log a}\le 2\frac{\log(32a^{5/2})}{\log a}\le 15,\quad\text{whenever } a\ge\lambda\ge 2.$$

□

By Proposition 2.5(a), a sequence of vertices $((x_i,n_i))_{i\in\mathbb{N}}\in\mathcal{S}$ converges at infinity if and only if $\lim n_i=\infty$ and $(x_i)_{i\in\mathbb{N}}$ is a convergent sequence in (X,d). Two sequences $((x_i,n_i))_{i\in\mathbb{N}}$ and $((y_i,m_i))_{i\in\mathbb{N}}$ that converge at infinity are equivalent if and only if $\lim_{i\to\infty}x_i=\lim_{i\to\infty}y_i$ and $\lim_{i\to\infty}n_i=\lim_{i\to\infty}m_i=\infty$. Define the *limit at infinity* function $\mathfrak{l}_\infty:\partial(\mathcal{S},D_1)\to X$ that maps an equivalence class of sequence converging at infinity to its *limit* as

$$\mathfrak{l}_\infty(((x_i,n_i))_{i\in\mathbb{N}})=\lim_{i\to\infty}x_i. \tag{2.5}$$

Note that $\mathfrak{l}_\infty$ is well defined and is a bijection. By the first estimate in Proposition 2.5(a), d is a visual metric on $\partial(\mathcal{S},D_1)$ with visual parameter a and base point v_0 in the following sense:

$$\frac{a-1}{4\lambda a}d(x,y)\le a^{-(\mathfrak{l}_\infty^{-1}(x)|\mathfrak{l}_\infty^{-1}(y))_1}\le\frac{a^{5/2}}{\lambda-1}d(x,y)\quad\text{for all } x,y\in X. \tag{2.6}$$

We would like to use Proposition 2.3 to estimate Gromov product in the hyperbolic filling. Since $(\mathcal{S},D_1)$ is not a geodesic space, we embed it into a geodesic space by replacing each edge with an isometric copy of the unit interval to obtain a metric space $(\widetilde{\mathcal{S}}, D_1)$ where we view $\mathcal{S} \subset \widetilde{\mathcal{S}}$ and D_1 on $\widetilde{\mathcal{S}}$ is an extension of D_1 on $\mathcal{S}$. For any $x, y \in X$, let $\widetilde{n}$ is the largest integer that satisfies $\{x, y\} \subset B(\widetilde{z}, 2a^{-\widetilde{n}})$ for some $(\widetilde{z}, \widetilde{n}) \in \mathcal{S}$ and define

$$c(x, y) = \{(\widetilde{z}, \widetilde{n}) \in \mathcal{S} : \{x, y\} \subset B(\widetilde{z}, 2a^{-\widetilde{n}})\}. \tag{2.7}$$

We think of $c(x, y)$ as the set of approximate centers of the triangle $[v_0, \mathfrak{l}_\infty^{-1}(x), \mathfrak{l}_\infty^{-1}(y)]$, where $\mathfrak{l}_\infty^{-1}(x), \mathfrak{l}_\infty^{-1}(y) \in \partial(\mathcal{S}, D_1)$ and v_0 is the unique element of $\mathcal{S}_0$. The following lemma makes this precise by identifying $c(x, y)$ as approximate centers of certain geodesic triangles in $\widetilde{\mathcal{S}}$.

Lemma 2.6. *Let the parameters of the hyperbolic filling satisfy*

$$a \geq \lambda \geq 2, \quad \textit{and} \quad \lambda \geq 1 + \frac{a}{a-1}.$$

Let $(z, m), (w, n) \in \mathcal{S}$ such that $x \in B(z, a^{-m}), y \in B(w, a^{-n})$ and such that $a^{-m} + a^{-n} < a^{-2} d(x, y)$. Then any $(w, k) \in c(x, y)$ is a K-approximate center for any geodesic triangle $[v_0, (z, m), (w, n)]$ in $(\widetilde{\mathcal{S}}, D_1)$, where $K = 80$.

Proof. Since every point in $\widetilde{\mathcal{S}}$ is at most distance $\frac{1}{2}$ away from a point in $\mathcal{S}$, by replacing points in $\widetilde{\mathcal{S}}$ with the corresponding closest points in $\mathcal{S}$, we obtain that $(\widetilde{\mathcal{S}}, D_1)$ is $(\delta + 3)$-hyperbolic whenever $(\mathcal{S}, D_1)$ is δ-hyperbolic.

Let $(\widetilde{z}, k) \in c(x, y)$. Since $\{x, y\} \subset B(\widetilde{z}, 2a^{-k})$, we obtain $d(x, y) \leq 4a^{-k}$. Choose $w \in X$ such that $(w, k+1) \in \mathcal{S}_{k+1}$ and $d(x, w) < a^{-(k+1)}$. By the maximality of k, we have that $y \notin B(w, 2a^{-(k+1)})$ and hence $d(x, y) \geq d(w, y) - d(w, x) > a^{-(k+1)}$. In particular,

$$a^{-(k+1)} < d(x, y) \leq 4a^{-k}. \tag{2.8}$$

If $a \geq 2$, we have $a^{-(k+1)} < d(x, y) \leq 4a^{-n} \leq a^{-k+2}$, which implies

$$-1 \leq k + \frac{\log d(x, y)}{\log a} \leq 2, \quad \text{whenever } a \geq 2. \tag{2.9}$$

Since $a^{-m} < a^{-2} d(x, y)$, we have $m + \frac{\log d(x,y)}{\log a} > 2$ which along with (2.9) implies that $m \geq k$. Choose (w, k) such that (z, m) is a descendant of (w, k). By Lemma 2.4(a), we have $d(x, w) \leq d(x, z) + d(w, z) < a^{-m} + \frac{a}{a-1} a^{-k} \leq \left(1 + \frac{a}{a-1}\right) a^{-k}$. Therefore if $\lambda \geq 1 + \frac{a}{a-1}$, we have $x \in B(w, \lambda a^{-k}) \cap B(\widetilde{z}, \lambda a^{-k}) \neq \emptyset$. Hence (w, k) and $(\widetilde{z}, k)$ are either equal or horizontal neighbors.

Note that $|d(x, y) - d(z, w)| \leq a^{-m} + a^{-n} < a^{-2} d(x, y)$, which implies

$$(1 - a^{-2}) d(x, y) \leq d(z, w) + a^{-m} + a^{-n} \leq (1 + 2a^{-2}) d(x, y).$$

Therefore if $a \geq \lambda \geq 2$, we have

$$\left| \frac{\log(d(x, y))}{\log a} - \frac{\log(d(z, w) + a^{-m} + a^{-n})}{\log a} \right| \leq 1.$$

Combining with (2.9) and Proposition 2.5(a), we obtain that

$$|((z,m)|(w,n))_1 - k| \leq 7.$$

This along with Proposition 2.3(a) and the fact that $(\widetilde{z},k)$ is a neighbor of (w,k), we obtain that $(\widetilde{z},k)$ is K-approximate center of the geodesic triangle $[v_0,(z,m),(w,n)]$, where $K = 1 + 7 + 4(15+3) = 80$. □

Remark 2.7. The assumption

$$a \geq \lambda \geq 6 \tag{2.10}$$

implies the estimates assumed on a, λ in Lemma 2.4, Proposition 2.5, and Lemma 2.6 hold. For this reason we assume (2.10) for much of this work. The analogous estimate [Car, (2.8)] is more complicated because it involves the constant in the definition of uniform perfectness.

3 Construction of metric and homogeneous measure

In this section, we construct metric in the conformal gauge and a homogeneous measure using a weight function on the hyperbolic filling $\mathcal{S}$ as constructed in §2.3. A *weight* on a filling $\mathcal{S}$ is a function $\rho : \mathcal{S} \to (0,\infty)$. We recall the definition of homogeneous measure and its relevance to Assouad dimension in §3.1. We then introduce and recall some hypothesis on a weight function on the hyperbolic filling that provides upper bound on $\dim_{\mathrm{CA}}(X,d)$ in §3.2.

3.1 Vol'berg-Konyagin theorem

Our approach to obtain upper and lower bounds on the conformal Assouad dimension ($\dim_{\mathrm{CA}}(X,d) \leq \mathrm{CE}(X,d)$ and $\dim_{\mathrm{CA}}(X,d) \geq \mathrm{CE}(X,d)$) relies on a theorem of Vol'berg and Konyagin that we recall below in Theorem 3.1. This result clarifies the relationship between Assouad dimension and doubling measures. A non-zero Borel measure μ on a metric space (X,d) is said to be *doubling* if there exists $C_D > 1$ such that

$$\mu(B(x,2r)) \leq C_D \mu(B(x,r)) \quad \text{for all } x \in X, r > 0,$$

where $B(x,r) = \{y \in X : d(x,y) < r\}$ denotes the open ball of radius r centered at x. A non-zero Borel measure is said to be *q-homogeneous measure* if there exists $C > 1$ such that

$$\mu(B(x,R)) \leq C\left(\frac{R}{r}\right)^q \mu(B(x,r)), \quad \text{for all } x \in X, 0 < r \leq R.$$

It is evident that a measure is doubling if and only if it is q-homogeneous for some $q \in (0,\infty)$. The fundamental relationship between Asssouad dimension and doubling measures is given by the following theorem of Vol'berg and Konyagin [VK].

Theorem 3.1. *[VK, Theorem 1] The Assouad dimension of compact metric space (X,d) is given by*

$$\dim_{\mathrm{A}}(X,d) = \inf\{q > 0 : \textit{there exists a } q\textit{-homogeneous measure on } (X,d)\}.$$

3.2 Weights on the filling and Carrasco-type hypotheses

Let $\pi_1 : \mathcal{S} \to X, \pi_2 : \mathcal{S} \to \mathbb{N}_{\geq 0}$ denote the projection maps such that $v = (\pi_1(v), \pi_2(v))$ for any $v \in \mathcal{S}$. We say that an edge between two vertices v and w is *horizontal* if $\pi_2(v) = \pi_2(w)$. For a vertex $v \in \mathcal{S}$, by B_v we denote the metric ball $B(\pi_1(v), a^{-\pi_2(v)})$. Given a vertex $v \in \mathcal{S}$, we define the *genealogy* $g(v)$ as a sequence of vertices $(v_0, v_1, \dots, v_k)$ where $v_k = v$ and v_i is the parent of v_{i+1} for all $i = 0, \dots, k-1$ and v_0 is the unique vertex in $\mathcal{S}_0$. Given a function $\rho : \mathcal{S} \to (0, \infty)$, we define $\pi : \mathcal{S} \to (0, \infty)$ as

$$\pi(v) = \prod_{w \in g(v)} \rho(w). \tag{3.1}$$

A *path* γ in $(\mathcal{S}, D_2)$ is a sequence of vertices $\gamma = (w_1, \dots, w_n)$ where there is an edge between w_i and w_{i+1} (that is, $D_2(w_i, w_{i+1}) = 1$) for each $i = 1, \dots, n-1$. The ρ-length of a path γ is defined by

$$L_\rho(\gamma) = \sum_{v \in \gamma} \pi(v). \tag{3.2}$$

The following two families of paths will play an important role in this work. A path is said to be *horizontal* if it only consists of horizontal edges. Given $x, y \in X$ and $n \in \mathbb{N}_{\geq 0}$, we define

$$\Gamma_n(x,y) = \left\{ \gamma = (v_1, \dots, v_k) \;\middle|\; \begin{array}{l} \gamma \text{ is a path in } (\mathcal{S}, D_2),\ \pi_2(v_1) = \pi_2(v_k) = n, \\ \text{and } x \in B_{v_1}, y \in B_{v_k}, k \in \mathbb{N} \end{array} \right\}. \tag{3.3}$$

For a vertex $v \in \mathcal{S}_k$, we define

$$\Gamma_k(v) = \inf \left\{ \gamma = (v_1, v_2, \dots, v_n) \;\middle|\; \begin{array}{l} \gamma \text{ is a horizontal path with } \pi_2(v_i) = k+1 \text{ for all } i, \\ n \in \mathbb{N},\ \pi_1(v_1) \in B_v, \pi_1(v_n) \notin 2 \cdot B_v \end{array} \right\}. \tag{3.4}$$

For $x, y \in X$, we define

$$\pi(c(x,y)) = \max\{\pi(w) : w \in c(x,y)\}, \tag{3.5}$$

where $c(x,y)$ is as defined in Lemma 2.6. We recall the Carrasco-type conditions imposed on the weight function $\rho : \mathcal{S} \to (0,1)$.

Assumption 3.2. A weight function $\rho : \mathcal{S} \to (0,1)$ may satisfy some of the following hypotheses:

(H1) There exist $0 < \eta_- \leq \eta_+ < 1$ so that $\eta_- \leq \rho(v) \leq \eta_+$ for all $v \in \mathcal{S}$.

(H2) There exists a constant $K_0 \geq 1$ such that for all $v, w \in \mathcal{S}$ that share a horizontal edge, we have

$$\pi(v) \leq K_0 \pi(w),$$

where π is as defined in (3.1).

(H3) There exists a constant $K_1 \geq 1$ such that for any pair of points $x, y \in X$, there exists $n_0 \geq 1$ such that if $n \geq n_0$ and γ is a path in $\Gamma_n(x, y)$, then

$$L_\rho(\gamma) \geq K_1^{-1}\pi(c(x, y)),$$

where $\Gamma_n(x, y), L_\rho, \pi(c(x, y))$ are as defined in (3.3), (3.2), and (3.5) respectively.

(H4) There exists $p > 0$ such that for all $v \in \mathcal{S}_m$ and $n > m$, we have

$$\sum_{w \in \mathcal{D}_n(v)} \pi(w)^p \leq \pi(v)^p,$$

where $\mathcal{D}_n(v)$ denotes the descendants of v in $\mathcal{S}_n$. Clearly, it suffices to impose the above condition only for $n = m + 1$.

($\widetilde{\mathrm{H4}}$) There exists $C > 1, p > 0$ such that for all $v \in \mathcal{S}_m$, and $n > m$, we have

$$C^{-1}\pi(v)^p \leq \sum_{w \in \mathcal{D}_n(v)} \pi(w)^p \leq C\pi(v)^p,$$

where $\mathcal{D}_n(v)$ denotes the descendants of v in $\mathcal{S}_n$.

The main results of this section are Theorems 3.3 and 3.4 which provide an upper bound on $\dim_{\mathrm{CA}}(X, d)$ under certain assumptions on the weight function on a filling. Theorem 3.3 is an analogue of [Car, Theorem 1.1] but the hypothesis (H4) is different from that of [Car, Theorem 1.1] where the hypothesis ($\widetilde{\mathrm{H4}}$) was used instead. As explained in the introduction, [Sha, Lemma 6.2] implies that (H1) along with ($\widetilde{\mathrm{H4}}$) can hold only on a uniformly perfect metric space. Since we consider metric spaces that are not necessarily uniformly perfect, we need to modify ($\widetilde{\mathrm{H4}}$) to (H4) as above. This hypothesis plays a key role in the upper bound on Assouad dimension. Another distinction from [Car] is that the weights can be used to construct essentially all metrics in $\mathcal{J}(X, d)$. Since our construction of metric relies on Proposition 2.2(c), we can only construct metrics in $\mathcal{J}_p(X, d)$ and hence we cannot obtain such a result. The following theorem is a counterpart of [Car, Theorem 1.1].

Theorem 3.3. *Let (X, d) be a compact doubling metric space and let $\mathcal{S}$ be a hyperbolic filling with vertical and horizontal parameters a, λ such that $a \geq \lambda \geq 6$. Let $\rho : \mathcal{S} \to (0, 1)$ be a weight function satisfying the hypothesis (H1), (H2), (H3), and (H4). Then there exists $\Theta_\rho \in \mathcal{J}_p(X, d)$ such that $\dim_{\mathrm{A}}(X, \Theta_\rho) \leq p$.*

The following is an analogue of [Car, Theorem 1.2]. The conclusion of Theorem 3.4 is the same as that of Theorem 3.3 but the hypotheses (H1), (H2), (H3), and (H4) are replaced by simpler hypotheses (S1) and (S2). The hypotheses (S1) and (S2) below are identical to [Car, Theorem 1.2] but in the conclusion we bound $\dim_{\mathrm{CA}}$ instead of $\dim_{\mathrm{ARC}}$.

Theorem 3.4. *Let (X, d) be a compact doubling metric space and let $\mathcal{S}$ denote a hyperbolic filling with vertical and horizontal parameters a, λ respectively such that $a \geq \lambda \geq 6$. Let $p > 0$. There exists $\eta_0 \in (0, 1)$ which depends only on p, λ and the doubling constant of (X, d) (but not on the vertical parameter a) such that if there exists a function $\sigma : \mathcal{S} \to [0, \infty)$ that satisfies:*

(S1) *for all* $v \in \mathcal{S}_k$ *and* $k \geq 0$, *if* $\gamma \in \Gamma_k(v)$, *then*

$$\sum_{w \in \gamma} \sigma(v) \geq 1,$$

where Γ_k *is as defined in* (3.4), *and*

(S2) *for all* $k \geq 0$ *and all* $v \in \mathcal{S}_k$, *we have*

$$\sum_{w \in C(v)} \sigma(v)^p \leq \eta_0,$$

then there exists $\Theta \in \mathcal{J}(X,d)$ *such that* $\dim_{\mathrm{A}}(X,\Theta) \leq p$ *and* $\Theta \in \mathcal{J}_p(X,d)$. *In particular,* $\dim_{\mathrm{A}}(X,d) \leq \inf\{\dim_{\mathrm{A}}(X,\theta) : \theta \in \mathcal{J}_p(X,d)\} \leq p$.

3.3 Construction of metric using weights on the filling

In this subsection, given a weight function which satisfies the hypotheses (H1), (H2), (H3), we construct a metric $\Theta_\rho \in \mathcal{J}_p(X,d)$ (Corollary 3.7). The main idea is to use weight ρ to induce a quasi-isometric change of metric on $\mathcal{S}$ (Lemma 3.5(a)) which in turn induces a power quasisymmetric change of metric on its boundary by Proposition 2.2(c). Since the boundary of $\mathcal{S}$ can be identified with (X,d) by Proposition 2.5(c), we therefore obtain a metric $\Theta_\rho \in \mathcal{J}_p(X,d)$. We remark that the hypothesis (H4) will not play any role in this subsection but will play a central role in the next one.

The weight function ρ which satisfies the hypotheses (H1)-(H3) induces a metric D_ρ on $\mathcal{S}$. We set the length of an edge $e=(v,w)$ as $\ell_\rho(e)$, where

$$\ell_\rho(e) = \begin{cases} 2\max\{-\log\eta_-, (-\log\eta_+)^{-1}, \log K_0\} & \text{if } e \text{ is horizontal,} \\ \log\frac{1}{\rho(v)} & \text{if } w \text{ is the parent of } v, \end{cases}$$

where η_-, η_+, K_0 are as defined in (H1) and (H2). This defines a metric

$$D_\rho(v,w) = \inf_\gamma \sum_{e \in \gamma} \ell_\rho(e), \tag{3.6}$$

where γ varies over all paths in the graph $(\mathcal{S}, D_2)$ from v to w and e varies over all edges in γ. By replacing each edge e with an isometric copy of the interval $[0, \ell_\rho(e)]$, we define a geodesic metric space $(\widetilde{\mathcal{S}}, D_\rho)$ such that $\mathcal{S} \subset \widetilde{\mathcal{S}}$ and the restriction of D_ρ on $\widetilde{\mathcal{S}}$ coincides with that of $\mathcal{S}$.

Lemma 3.5. *Let* (X,d) *and* $\mathcal{S}$ *be as in the statement of Theorem 3.3. Let* $\rho : \mathcal{S} \to (0,\infty)$ *which satisfies hypotheses* (H1) *and* (H2). *Let* D_ρ *denote the metric defined in* (3.6).

(a) $(\mathcal{S}, D_\rho)$ *is approximately geodesic. The identity map* $\mathrm{Id} : (\mathcal{S}, D_1) \to (\mathcal{S}, D_\rho)$ *is a quasi-isometry.*

(b) *Any path of the form $((x_i,n_i))_{1\le i\le k}$ such that (x_i,n_i) is the parent of (x_{i+1},n_{i+1}) for all $i=1,\dots,k-1$ defines a shortest path in the D_ρ metric and hence*

$$D_\rho((x_1,n_1),(x_k,n_k)) = \left|\log\frac{1}{\pi((x_1,n_1))} - \log\frac{1}{\pi((x_k,n_k))}\right|.$$

(c) *A sequence of vertices converges at infinity in $(\mathcal{S},D_1)$ if and only if it converges at infinity $(\mathcal{S},D_\rho)$. Two sequences that converge at infinity are equivalent in $(\mathcal{S},D_1)$ if and only if they are equivalent in $(\mathcal{S},D_\rho)$. In particular, the identity map is a well-defined bijection between $\partial(\mathcal{S},D_1)$ and $\partial(\mathcal{S},D_\rho)$. Therefore the limit at infinity map $\mathfrak{l}_\infty : \partial(\mathcal{S},D_1)\to X$ defined in (2.5) is also well-defined as $\mathfrak{l}_\infty : \partial(\mathcal{S},D_\rho)\to X$. Furthermore, there exists $C>1$ such that the Gromov product satisfies*

$$C^{-1}C\pi(c(x,y)) \le e^{-(\mathfrak{l}_\infty^{-1}(x)|\mathfrak{l}_\infty^{-1}(y))_\rho} \le C\pi(c(x,y)) \quad \textit{for all } x,y\in X,$$

where $(\cdot|\cdot)_\rho$ is the Gromov product on $(\mathcal{S},D_\rho)$ with base point $v_0\in\mathcal{S}_0$.

(d) *There exists $C>0$ such that the following holds: for any pair of distinct points $x,y\in X$, there exists n_0 such that whenever $n\ge n_0$ and $u,v\in\mathcal{S}_n$ such that $x\in B_u, y\in B_v$, there exists a path $\gamma=(w_i)_{i=0,\dots,k}$ in the graph $(\mathcal{S},D_2)$ such that $L_\rho(\gamma)\le C\pi(c(x,y))$.*

Proof.

(a) It is easy to check that $(\mathcal{S},D_\rho)$ is $2\max\{-\log\eta_-,(-\log\eta_+)^{-1},\log K_0\}$-approximately geodesic, since the horizontal edges are the longest edges. The fact that the identity map is a quasi-isometry is because there exist constants C_1,C_2 such that $C_1\le \ell_\rho(e)\le C_2$ for all edges e. This along with Lemma 2.4(d) implies that $C_1D_1\le C_1D_2\le D_\rho\le C_2D_2\le 2C_2D_1$.

(b) This follows from the same argument as [Car, Proof of Lemma 2.3] which uses Lemma 2.4(b).

(c) The first three claims follow from Proposition 2.2. Let $(v_n)_{n\in\mathbb{N}},(w_n)_{n\in\mathbb{N}}$ be two sequences of vertices such that $x\in B_{v_n}, y\in B_{w_n}, v_n\in\mathcal{S}_n, w_n\in\mathcal{S}_n$ for all $n\in\mathbb{N}$. By Lemma 2.6, every vertex in $c(x,y)$ is a 80-approximate center for the geodesic triangle $[v_0,v_n,w_n]$ (in $(\widetilde{\mathcal{S}},D_1)$) for all large enough n. By (a) and Proposition 2.3(d), every vertex in $c(x,y)$ is a K'-approximate center of the geodesic triangle $[v_0,v_n,w_n]$ (in $(\widetilde{\mathcal{S}},D_\rho)$) for some $K'>0$. By Proposition 2.3(d), we obtain the desired estimate.

(d) Let $x,y\in X$ and $u,v\in\mathcal{S}_n$ be as in the statement of the lemma. We choose n_0 be the integer such that $c(x,y)\subset\mathcal{S}_{n_0}$. Let $\widetilde{u},\widetilde{v}\in\mathcal{S}_{n_0}$ be the vertices such that u,v are descendants of $\widetilde{u},\widetilde{v}$ respectively. Let $\widetilde{c}\in c(x,y)$. As shown in the proof of Lemma 2.6, either $\widetilde{u}$ (resp. $\widetilde{v}$) is equal to $\widetilde{c}$ or is a horizontal neighbor of $\widetilde{u}$ (resp. $\widetilde{v}$). Therefore by (H2), $\pi(\widetilde{u})\vee\pi(\widetilde{v})\le K_0\pi(c(x,y))$. We now construct the desired path γ from u to v as follows. We join u to $\widetilde{u}$ and $\widetilde{v}$ to v using the geneology. We can connect $\widetilde{u}$ to $\widetilde{v}$ using $\widetilde{c}$ if necessary. By (H1) and (H2), the length $L_\rho(\gamma)$ of γ is bounded by

$$L_\rho(\gamma)\le \pi(c(x,y))\left(1+\frac{2K_0}{1-\eta_+}\right).$$

$\square$

The following proposition provides a bound on visual parameter on $\partial(\mathcal{S}, D_\rho)$ and relies crucially on (H3). This construction of metric is slightly different from that of [Car] and [Sha, Theorem 5.1].

Proposition 3.6 (Visual parameter control). *Let (X, d) and $\mathcal{S}$ be as in the statement of Theorem 3.3. Let $\rho : \mathcal{S} \to (0, \infty)$ which satisfies hypotheses (H1), (H2), and (H3). Then there exists a visual metric θ_ρ on $\partial(\mathcal{S}, D_\rho)$ with base point $v_0 \in \mathcal{S}_0$ with visual parameter e, where D_ρ is as defined in (3.6). There exists $C_1 > 1$ such that the metric θ_ρ satisfies*

$$C_1^{-1}\pi(c(x, y)) \le \theta_\rho(\iota_\infty^{-1}(x), \iota_\infty^{-1}(y)) \le \pi(c(x, y)) \quad \textit{for all } x, y \in X.$$

Furthermore, the map $p : (\partial(\mathcal{S}, D_\rho), \theta_\rho) \to (X, d)$ is a power quasisymmetry.

Proof. We define the desired metric θ_ρ as

$$\theta_\rho(\iota_\infty^{-1}(x), \iota_\infty^{-1}(y)) = \inf\left\{\sum_{i=0}^{k-1} \pi(c(x_i, x_{i+1})) : k \in \mathbb{N}, x_0 = x, x_k = y, x_i \in X \text{ for all } i\right\}.$$

Clearly, θ_ρ is non-negative, satisfies the triangle inequality and $\theta_\rho(\iota_\infty^{-1}(x), \iota_\infty^{-1}(y)) \le \pi(c(x, y))$ for all $x, y \in X$. It suffices to show

$$\theta_\rho(\iota_\infty^{-1}(x), \iota_\infty^{-1}(y)) \gtrsim \pi(c(x, y))$$

To this end consider a sequence $x_0, \ldots, x_k$ such that $x_0 = x, x_k = y$. Without loss of generality, we may assume that $x_i \ne x_{i+1}$ for all $i = 0, \ldots, k-1$. We choose n large enough so that we can apply Lemma 3.5(d) to each pair $x_i, x_{i+1}, i = 0, \ldots, k-1$. Choose $v_i \in \mathcal{S}_n$ such that $x_i \in B_{v_i}$ for all $i = 0, \ldots, k$. By concatenating all points obtained by applying Lemma 3.5(d) to each pair x_i, x_{i+1}, we obtain a path $\gamma \in \Gamma_n(x, y)$ such that

$$L_\rho(\gamma) \le C \sum_{i=0}^{k-1} \pi(c(x_i, x_{i+1})),$$

where C is the constant from Lemma 3.5(d). Combining the above estimate with (H3), we obtain

$$(K_1 C)^{-1}\pi(c(x, y)) \le \theta_\rho(\iota_\infty^{-1}(x), \iota_\infty^{-1}(y)) \le \pi(c(x, y)), \tag{3.7}$$

where K_1 is the constant in (H3).

By Proposition 2.5(c), the map $p : (\partial(\mathcal{S}, D_1), \theta_1) \to (X, d)$ is a bi-Lipschitz map where θ_1 is a visual metric on $\partial(\mathcal{S}, D_1)$ with base point $v_0 \in \mathcal{S}_0$ and visual parameter a. Since the identity map $\mathrm{Id} : (\mathcal{S}, D_\rho) \to (\mathcal{S}, D_1)$ is a quasi-isometry by Lemma 3.5(a), the induced boundary map (as defined in Proposition 2.2(b)) $\partial\,\mathrm{Id} : (\partial(\mathcal{S}, D_\rho), \theta_\rho) \to (\partial(\mathcal{S}, D_1), \theta_1)$ is a power quasisymmetry by Proposition 2.2(c). Composing this power quasisymmetry $\partial\,\mathrm{Id} : (\partial(\mathcal{S}, D_\rho), \theta_\rho) \to (\partial(\mathcal{S}, D_1), \theta_1)$ with the bi-Lipschitz map $p : (\partial(\mathcal{S}, D_1), \theta_1) \to (X, d)$

yields the desired conclusion that $p : (\partial(\mathcal{S}, D_\rho), \theta_\rho) \to (X, d)$ is a power quasisymmetry. $\square$

Consider the metric $\Theta_\rho : X \times X \to [0, \infty)$ defined by

$$\Theta_\rho(x, y) := \theta_\rho(\iota_\infty^{-1}(x), \iota_\infty^{-1}(y)). \tag{3.8}$$

where θ_ρ is the visual metric from Proposition 3.6. The following is an immediate corollary of Proposition 3.6.

Corollary 3.7. *Let (X, d) and $\mathcal{S}$ be as in the statement of Theorem 3.3. Let $\rho : \mathcal{S} \to (0, \infty)$ which satisfies hypotheses (H1), (H2), and (H3). Then the metric Θ_ρ defined in (3.8) satisfies $\Theta_\rho \in \mathcal{J}_p(X, d)$.*

The following lemma provides a sequence in $\iota_\infty^{-1}(x) \in \partial(\mathcal{S}, D_2)$ with desirable properties and is useful for approximating balls centered at $x \in X$ in different metrics.

Lemma 3.8. *Let (X, d) be a compact doubling metric space and let $(\mathcal{S}, D_2)$ denote the corresponding hyperbolic filling with vertical and horizontal parameters a, λ respectively such that $a \ge \lambda \ge 6$. For any $x \in X$, there exists a sequence of vertices $v_n \in \mathcal{S}_n, n \ge 0$ such that v_n is the parent of v_{n+1} and $d(x, \pi_1(v_n)) \le \frac{1}{1-a^{-1}} a^{-n}$ for all $n \in \mathbb{N}_{\ge 0}$.*

Proof. For each n, we choose $w_n \in \mathcal{S}_n$ be such that $d(w_n, x) < a^{-n}$ (this is possible since X_n is a maximal a^{-n}-separated subset). Consider the sequence of genealogies $g(w_n)$ for each $n \in \mathbb{N}_{\ge 0}$. By a diagonal argument the sequence of genealogies $g(w_n)$ converge along a subsequence to yield the a sequence $v_n \in \mathcal{S}_n$ such that v_n is the parent of v_{n+1} for all $n \in \mathbb{N}_{\ge 0}$. If $v_n \in \mathcal{S}_n$ is in the genealogy of $w_k, k > n$, we have $d(x, \pi_1(v_n)) < a^{-k} + \sum_{i=n}^{k} a^{-k}$. Letting $k \to \infty$ along a subsequence yields the desired bound $d(x, \pi_1(v_n)) \le \frac{1}{1-a^{-1}} a^{-n}$. $\square$

The following lemma provides an approximation of balls in (X, Θ_ρ) using the balls in (X, d). In the following lemma, we use the notation $B_\Theta(\cdot,\cdot), B_d(\cdot,\cdot)$ to denote the balls in the metrics Θ_ρ, d respectively.

Lemma 3.9. *Let (X, d) and $\mathcal{S}$ be as in the statement of Theorem 3.3. Let $\rho : \mathcal{S} \to (0, \infty)$ which satisfies hypotheses (H1), (H2), and (H3). Let $\Theta_\rho \in \mathcal{J}_p(X, d)$ be as defined in (3.8) and let K_0 denote the constant in (H2). Let $L > 1$ such that*

$$\frac{1}{L}\pi(c(x, y)) \le \Theta_\rho(x, y) \le \pi(c(x, y)) \quad \textit{for all } x, y \in X. \tag{3.9}$$

For any $x \in X$, let $(v_n)_{n \in \mathbb{N}_{\ge 0}}$ denote a sequence such that $v_n \in \mathcal{S}_n$ for all $n \in \mathbb{N}_{\ge 0}$ as given in Lemma 3.8. Then

$$B_d(\pi_1(v_k), 2a^{-k}) \subset B_\Theta(x, r), \quad \textit{whenever } v_k \textit{ satisfies } \pi(v_k) < K_0^{-1} r, \tag{3.10}$$

and

$$B_\Theta(x, r) \subset B_d(\pi_1(v_k), 2(\lambda + 2)a^{-k}), \tag{3.11}$$

whenever $k \in \mathbb{N}$ is such that $\pi(v_k) \le K_0 L r$ and $\pi(v_{k-1}) > K_0 L r$.

Proof. First, we show (3.10). Let $k \in \mathbb{N}_{\geq 0}$ be such that $\pi(v_k) < K_0^{-1} r$ and let $y \in B_d(\pi_1(v_k), 2a^{-k})$. Let $(z,l) \in c(x,y)$. Since $\{x,y\} \subset B_d(\pi_1(v_k), 2a^{-k})$, we have $l \leq k$. Note that $D_2((z,l), v_l) \leq 1$, since $d(\pi_1(v_l), x) \leq d(\pi_1(v_l), \pi_1(v_k)) + d(\pi_1(v_k), x) \leq (1 - a^{-1})^{-1} a^{-l} + (1-a^{-1})^{-1} a^{-k} \leq 2(1-a^{-1})^{-1} a^{-l} < \lambda a^{-l}$ and $x \in B(z, 2a^{-1}) \subset B(z, \lambda a^{-l})$. This along with (H2) implies that

$$\pi(c(x,y)) \leq K_0 \pi(v_l) \leq K_0 \pi(v_k) < r.$$

This estimate along with (3.9) implies (3.10).

Next, we show (3.11). Let $k \in \mathbb{N}$ be such that $\pi(v_k) \leq K_0 L r$ and $\pi(v_{k-1}) > K_0 L r$ and let $y \in B_\Theta(x,r)$. By (3.9), $\pi(c(x,y)) \leq Lr$. Let $(z,l) \in c(x,y)$. Note that $D_2(v_l, (z,l)) \leq 1$ which along with (H2) implies that

$$\pi(v_l) \leq K_0 \pi(c(x,y)) \leq K_0 L r.$$

The choice of k implies that $l \geq k$. Hence by Lemma 3.8

$$\begin{aligned} d(\pi_1(v_k), y) &\leq d(\pi_1(v_k), \pi_1(v_l)) + d(\pi_1(v_l), z) + d(z,y) \\ &< (1-a^{-1})^{-1} a^{-k} + 2\lambda a^{-l} + 2a^{-l} \quad (\text{since } D_2(v_l,(z,l)) \leq 1 \text{ and } (z,l) \in c(x,y)) \\ &\leq \left(2 + (1-a^{-1})^{-1} + 2\lambda\right) a^{-k} < 2(2+\lambda) a^{-k} \quad (\text{since } k \geq l). \end{aligned} \tag{3.12}$$

This completes the proof of (3.11). $\square$.

3.4 Construction of homogeneous measures using weights

Next, we need to control the Assouad dimension of (X, Θ_ρ), where $\Theta_\rho \in \mathcal{J}_p(X,d)$ is as given in Corollary 3.7. To this end, we construct a p-homogeneous measure on (X, Θ_ρ) using hypothesis (H4). This along with Theorem 3.1 implies an upper bound on the Assouad dimension $\dim_{\mathrm{A}}(X, \Theta_\rho) \leq p$. To this end, we construct a doubling measure on X using the weight function $\rho : \mathcal{S} \to (0,\infty)$. The idea is to construct a measure on X as a limit of discrete measures on $\mathcal{S}_k$ as $k \to \infty$. To this end, we introduce the following notions.

Definition 3.10. Let $k \in \mathbb{N}, p \in (0,\infty), C \in (1,\infty)$ and $f_0 : \mathcal{S}_k \to (0,\infty), f_1 : \mathcal{S}_{k+1} \to (0,\infty)$. Let $\pi : \mathcal{S} \to (0,\infty)$ be a weight.

(a) We say that $f_0 : \mathcal{S}_k \to (0,\infty)$ is (C,π)-*balanced* if $\frac{f_0(u)}{\pi(u)^p} \leq C^2 \frac{f_0(v)}{\pi(v)^p}$ for all vertices $u, v \in \mathcal{S}_k$ with $D_2(u,v) = 1$.

(b) A function $f_0 : \mathcal{S}_k \to (0,\infty)$ is (C,π)-*unbalanced on* $e = \{u,v\}$ if either $\frac{f_0(u)}{\pi(u)^p} > C^2 \frac{f_0(v)}{\pi(v)^p}$ or $\frac{f_0(v)}{\pi(v)^p} > C^2 \frac{f_0(u)}{\pi(u)^p}$. Similarly, we say that a function $f_0 : \mathcal{S}_k \to (0,\infty)$ is (C,π)-*balanced on* $e = \{u,v\}$ if $C^{-2} \frac{f_0(v)}{\pi(v)^p} \leq \frac{f_0(u)}{\pi(u)^p} \leq C^2 \frac{f_0(v)}{\pi(v)^p}$.

(c) We say that the pair (f_0, f_1) is (C,π)-*compatible* if for all points $u \in \mathcal{S}_k$ and $v \in \mathcal{S}_{k+1}$ such that u is the parent of v, we have

$$\frac{f_0(u)}{\pi(u)^p} \le \frac{f_1(v)}{\pi(v)^p} \le C\frac{f_0(u)}{\pi(u)^p}. \tag{3.13}$$

We remark that the notions of balanced and compatibility depend only on the horizontal and vertical edges of $(\mathcal{S}, D_2)$ respectively.

Given a horizontal edge $e = \{u,v\}$ in $\mathcal{S}_k$, we define the (C,π)-*balancing operator* $B_k^e : (0,\infty)^{\mathcal{S}_k} \to (0,\infty)^{\mathcal{S}_k}$ as follows. If f_0 is (C,π)-balanced on e, we set $B_k^e(f_0) \equiv f_0$. Otherwise, if $\frac{f_0(u)}{\pi(u)^p} > C^2\frac{f_0(v)}{\pi(v)^p}$ we set

$$(B_k^e(f_0))(w) = \begin{cases} f_0(w) & \text{if } w \notin \{u,v\}, \\ f_0(u) - \alpha_1 & \text{if } w = u, \\ f_0(v) + \alpha_1 & \text{if } w = v, \end{cases}$$

where α_1 is given by

$$\alpha_1\left(\frac{C^2}{\pi(v)^p} + \frac{1}{\pi(u)^p}\right) = \frac{f_0(u)}{\pi(u)^p} - C^2\frac{f_0(v)}{\pi(v)^p},$$

so that $\frac{(B_k^e(f_0))(u)}{\pi(u)^p} = C^2\frac{(B_k^e(f_0))(v)}{\pi(v)^p}$. The case $\frac{f_0(v)}{\pi(v)^p} > C^2\frac{f_0(u)}{\pi(u)^p}$ is similar to $\frac{f_0(u)}{\pi(u)^p} > C^2\frac{f_0(v)}{\pi(v)^p}$. The terminology is due to the fact that $B_k^e(f_0)$ is (C,π)-balanced on e for all $f \in (0,\infty)^{\mathcal{S}_k}$.

We need the following modification of a lemma of Vol'berg and Konyagin [VK, Lemma, p. 631] which plays a key role in the construction of doubling measures.

Lemma 3.11. *Let (X,d) and $\mathcal{S}$ be as in the statement of Theorem 3.3. Let $\rho : \mathcal{S} \to (0,\infty)$ be function that satisfies hypotheses (H1) and (H4). Let $C \ge \eta_-^{-p}$, where the constants η_-, p are as given in the hypotheses (H1) and (H4). Let $k \in \mathbb{N}_{\ge 0}$, and let μ_k be a probability mass function on $\mathcal{S}_k$ such that μ_k is (C,π)-balanced. Then there exists a probability mass function μ_{k+1} on $\mathcal{S}_{k+1}$ such that the following hold.*

(1) *The pair (μ_k, μ_{k+1}) is (C,π)-compatible.*

(2) *The function μ_{k+1} is (C,π)-balanced.*

(3) *The construction of the measure μ_{k+1} from the measure μ_k can be regarded as the transfer of masses from the points of X_k to those of X_{k+1}, with no mass transferred over a distance greater than $(1+2\lambda a^{-1})a^{-k}$. More precisely, there is a probability measure $\mu_{k,k+1}$ on $X \times X$ which is a coupling of the probability measures $\widetilde{\mu}_k := \sum_{u\in\mathcal{S}_k} \mu_k(u)\delta_{\pi_1(u)}, \widetilde{\mu}_{k+1} := \sum_{v\in\mathcal{S}_{k+1}} \mu_{k+1}(v)\delta_{\pi_1(v)}$ such that*

$$\mu_{k,k+1}\left(\{(x_1,x_2) \in X \times X : d(x_1,x_2) \ge (1+2\lambda a^{-1})a^{-k}\}\right) = 0,$$

where δ_x denotes the Dirac measure at $x \in X$. Here by a coupling we mean the projection maps from $X \times X$ to X on the first and second component pushes forward the measure $\mu_{k,k+1}$ to $\widetilde{\mu}_k$ and $\widetilde{\mu}_{k+1}$ respectively.

The proof of Lemma 3.11 is done in two steps. First is an 'averaging' step where we construct a measure on $\mathcal{S}_{k+1}$ by distributing the mass $\mu_k(u)$ of every vertex $u \in \mathcal{S}_k$ to its children so that the mass received by each child v is proportional to $\pi(v)^p$. At end of this step, we obtain a measure which satisfies the compatibility condition but not necessarily (C,π)-balanced. In the second 'balancing' step, we ensure that the measure is (C,π)-balanced by a repeated *local transfer* of mass along edges in $\mathcal{S}_{k+1}$ using the balancing operators B^e_{k+1}. By a *local* transfer we mean that the mass is transferred from a vertex to its neighbor. The next two lemmas show useful properties of the balancing operators. The first one shows that the compatibility condition is preserved by balancing operators.

Lemma 3.12. *Let (X,d) and $\mathcal{S}$ be as in the statement of Theorem 3.3. Let $\rho : \mathcal{S} \to (0,\infty)$ be function that satisfies hypotheses (H1) and (H4). Let μ_k be a probability mass function on $\mathcal{S}_k$ that is (C,π)-balanced for some $C > 1$. Let $f_0 : \mathcal{S}_{k+1} \to (0,1]$ be a probability mass function on $\mathcal{S}_{k+1}$ such that (μ_k, f_0) is (C,π)-compatible. Let $e = \{w_1, w_1'\}$ be an edge in $\mathcal{S}_{k+1}$ such that f_0 is (C,π)-unbalanced on e. Then the pair $(\mu_k, B^e_{k+1}(f_0))$ is also (C,π)-compatible.*

Proof. Without loss of generality, we assume that $\frac{f_0(w_1)}{\pi(w_1)^p} > C^2 \frac{f_0(w_1')}{\pi(w_1')^p}$. Let v_1 and v_1' be parents of w_1, w_1' respectively and let $f_1 := B^e_{k+1}(f_0)$. By construction, we have

$$f_1(w_1) < f_0(w_1), \quad f_1(w_1') > f_0(w_1'). \tag{3.14}$$

Therefore by the (C,π)-compatibility of (μ_k, f_0) and (3.14), we have

$$\frac{f_1(w_1)}{\pi(w_1)^p} \le C \frac{\mu_k(v_1)}{\pi(v_1)^p}, \quad \frac{f_1(w_1')}{\pi(w_1')^p} \ge \frac{\mu_k(v_1')}{\pi(v_1')^p}.$$

Therefore it suffices to verify that

$$\frac{f_1(w_1)}{\pi(w_1)^p} \ge \frac{\mu_k(v_1)}{\pi(v_1)^p}, \quad \frac{f_1(w_1')}{\pi(w_1')^p} \le C \frac{\mu_k(v_1')}{\pi(v_1')^p}. \tag{3.15}$$

Suppose the first inequality in (3.15) fails to be true, then by construction, (3.14) and the (C,π)-compatibility of (μ_k, f_0), we have

$$\frac{\mu_k(v_1)}{\pi(v_1)^p} > \frac{f_1(w_1)}{\pi(w_1)^p} = C^2 \frac{f_1(w_1')}{\pi(w_1')^p} > C^2 \frac{f_0(w_1')}{\pi(w_1')^p} \ge C^2 \frac{\mu_k(v_1')}{\pi(v_1')^p}, \tag{3.16}$$

which implies $\frac{\mu_k(v_1)}{\pi(v_1)^p} > C^2 \frac{\mu_k(v_1')}{\pi(v_1')^p}$. However, Lemma 2.4(b) implies that $D_2(v_1, v_1') \le 1$ and therefore the above estimate contradicts the assumption that μ_k is (C,π)-balanced. This proves the first inequality in (3.15). The proof of the second inequality in (3.15) is similar. Indeed, assume to the contrary that $\frac{f_1(w_1')}{\pi(w_1')^p} > C \frac{\mu_k(v_1')}{\pi(v_1')^p}$; then we have

$$\frac{\mu_k(v_1)}{\pi(v_1)^p} \ge C^{-1} \frac{f_0(w_1)}{\pi(w_1)^p} > C^{-1} \frac{f_1(w_1)}{\pi(w_1)^p} = C \frac{f_1(w_1')}{\pi(w_1')^p} > C^2 \frac{\mu_k(v_1')}{\pi(v_1')^p}, \tag{3.17}$$

which again implies $\frac{\mu_k(v_1)}{\pi(v_1)^p} > C^2 \frac{\mu_k(v_1')}{\pi(v_1')^p}$, a contradiction to the assumption that μ_k is (C,π)-balanced. In particular $(\mu_k, B^e_{k+1}(f_0))$ is (C,π)-balanced. $\square$

The next property is that a balancing operator cannot create unbalanced edges. More precisely, we have the following lemma.

Lemma 3.13. *Let (X,d) and $\mathcal{S}$ be as in the statement of Theorem 3.3. Let $\rho : \mathcal{S} \to (0,\infty)$ be function that satisfies hypotheses (H1) and (H4). Let μ_k be a probability mass function on $\mathcal{S}_k$ that is (C,π)-balanced for some $C > 1$. Let $f_0 : \mathcal{S}_{k+1} \to (0,1]$ be a probability mass function on $\mathcal{S}_{k+1}$ such that (μ_k, f_0) is (C,π)-compatible. Let $e = \{w_1, w_1'\}$ be an edge in $\mathcal{S}_{k+1}$ such that f_0 is (C,π)-unbalanced on e. If an edge $e' = \{w, w'\}$ on $\mathcal{S}_{k+1}$ is such that f_0 is (C,π)-balanced on e', then $B^e_{k+1}(f_0)$ is also (C,π)-balanced on e'.*

Proof. Without loss of generality, we assume that $\frac{f_0(w_1)}{\pi(w_1)^p} > C^2 \frac{f_0(w_1')}{\pi(w_1')^p}$. Let $e' = \{w, w'\}$ be such that f_0 is (C,π)-balanced on e'. Let $f_1 := B^e_{k+1}(f_0)$. By our assumption $\frac{f_0(w_1)}{\pi(w_1)^p} > C^2 \frac{f_0(w_1')}{\pi(w_1')}$, we have $\{w, w'\} \neq \{w_1, w_1'\}$. If $\{w, w'\} \cap \{w_1, w_1'\} = \emptyset$, then there is nothing to prove since f_0 and f_1 agree on $\{w, w'\}$.

The only remaining case to consider is if $\{w, w'\} \cap \{w_1, w_1'\}$ contains exactly one element. Next, we consider the case $\{w_1, w_1'\} \cap \{w, w'\} = \{w_1\}$ where $w_1 = w$. Since $f_0(w)/\pi(w)^p > C^2 f_0(w_1')/\pi(w_1')^p$,

$$\frac{f_1(w)}{\pi(w)^p} = \frac{f_1(w_1)}{\pi(w_1)^p} = C^2 \frac{f_1(w_1')}{\pi(w_1)^p}, \quad f_1(w) < f_0(w), \quad f_1(w') = f_0(w'). \tag{3.18}$$

We need to show that

$$\frac{f_1(w')}{\pi(w')^p} \le C^2 \frac{f_1(w)}{\pi(w)^p}, \quad \frac{f_1(w)}{\pi(w)^p} \le C^2 \frac{f_1(w')}{\pi(w')^p}. \tag{3.19}$$

Therefore by (3.18), only the first inequality in (3.19) can fail for f_1. Suppose that this happens, that is

$$\frac{f_1(w')}{\pi(w')^p} > C^2 \frac{f_1(w)}{\pi(w)^p}. \tag{3.20}$$

Let $v', v_1' \in \mathcal{S}_k$ be parents of w', w_1' respectively. By Lemma 3.12, (μ_k, f_1) is (C,π)-compatible. Then by the (C,π)-compatibility of (μ_k, f_1), (3.20), and (3.18), we obtain

$$\frac{\mu_k(v')}{\pi(v')^p} \ge C^{-1} \frac{f_1(w')}{\pi(w')^p} \overset{(3.20)}{>} C \frac{f_1(w)}{\pi(w)^p} \overset{(3.18)}{=} C^3 \frac{f_1(w_1')}{\pi(w_1')^p} \ge C^3 \frac{\mu_k(v_1')}{\pi(v_1')^p} > C^2 \frac{\mu_k(v_1')}{\pi(v_1')^p}, \tag{3.21}$$

which contradicts the assumption that μ_k is (C,π)-balanced (since $D_2(v', v_1') \le 1$ by Lemma 2.4(b) and $\lambda \ge 2 + 4\lambda a^{-1}$). The remaining case $\{w_1, w_1'\} \cap \{w, w'\} = \{w_1'\}$ is analyzed similarly and therefore the assertion that $B^e_{k+1}(f_0)$ is also (C,π)-balanced on e' is proved. $\square$

The following iterative construction uses Lemmas 3.12 and 3.13 to obtain a balanced and compatible function from a compatible function.

Lemma 3.14. *Let (X, d) and $\mathcal{S}$ be as in the statement of Theorem 3.3. Let $\rho : \mathcal{S} \to (0, \infty)$ be function that satisfies hypotheses (H1) and (H4). Let μ_k be a probability mass function on $\mathcal{S}_k$ that is (C, π)-balanced for some $C > 1$. Let $f_0 : \mathcal{S}_{k+1} \to (0, 1]$ be a probability mass function on $\mathcal{S}_{k+1}$ such that (μ_k, f_0) is (C, π)-compatible. Let $p_i = \{v_i, v_i'\}, i = 1, \ldots, T$ be an enumeration of all edges in $\mathcal{S}_{k+1}$. We inductively define*

$$f_i := B_{k+1}^{p_i}(f_{i-1}) \in (0, 1]^{\mathcal{S}_{k+1}} \quad \textit{for all } i = 1, \ldots, T. \tag{3.22}$$

Then, $f_i, i = 0, \ldots, T$ satisfy the following properties.

(a) Each f_i is a probability mass function such that (μ_k, f_i) is (C, π)-compatible for all $i = 0, 1, \ldots, T$.

(b) The probability mass function f_T is (C, π)-balanced.

(c) There are no pairs of edges $p_l = \{w_1, w_2\}$, $p_n = \{w_2, w_3\}$, $l, n \in \mathbb{Z} \cap [1, T], l < n$, such that mass is transferred from w_1 to w_2 in the transition from f_{l-1} to f_l and then mass is transferred from w_2 to w_3 in the transition from f_{n-1} to f_n.

Proof.

(a) Since the balancing operators preserve the sum, each f_i is a probability mass function. By Lemma 3.12, (μ_k, f_i) is (C, π)-compatible.

(b) This is an immediate consequence of Lemma 3.13, since f_T is (C, π)-balanced on every edge in $\mathcal{S}_{k+1}$.

(c) Assume the opposite; that is, there are a mass transfer from w_1 to w_2 (in the transition from f_{l-1} to f_l) followed by a mass transfer from w_2 to w_3 (in the transition from f_{n-1} to f_n with $n > l$), so that

$$\frac{f_l(w_1)}{\pi(w_1)^p} = C^2 \frac{f_l(w_2)}{\pi(w_2)^p}, \quad \frac{f_{n-1}(w_2)}{\pi(w_2)^p} > C^2 \frac{f_{n-1}(w_3)}{\pi(w_3)^p}. \tag{3.23}$$

By choosing l as the largest number less than n such that mass is transferred into w_2 in the transition from f_{l-1} to f_l before the mass transfer from w_2 to w_3 takes place in the transition from f_{n-1} to f_n, we may assume that

$$f_l(w_2) = f_{n-1}(w_2). \tag{3.24}$$

If v_1, v_3 denote the parents of w_1, w_3 respectively, then by Lemma 2.4(b)

$$D_2(v_1, v_3) \leq 1.$$

Consequently by assumption that μ_k is (C, π)-balanced, we have $\mu_k(v_1)/\pi(v_1)^p \leq C^2 \mu_k(v_3)/\pi(v_3)^p$. However (3.23), (C, π)-compatibility of $(\mu_k, f_l), (\mu_k, f_{n-1})$ along with (3.24) imply the opposite inequality $\mu_k(v_1)/\pi(v_1)^p > C^2 \mu_k(v_3)/\pi(v_3)^p$. We have arrived at the desired contradiction and therefore the property (c) is verified.

□

Next, we prove Lemma 3.11 by using the inductive construction in Lemma 3.14.

Proof of Lemma 3.11. Let $k \in \mathbb{N}_{\geq 0}$, and let μ_k be a (C, π)-balanced probability mass function on $\mathcal{S}_k$. As explained earlier, the transfer of mass is accomplished in two steps. In the first 'averaging' step, we distribute the mass $\mu_k(v)$ to all its children such that the mass distributed to each child w is proportional to $\pi(w)^p$ (or equivalently $\rho(w)^p$); that is

$$f_0(w) = \frac{\pi(w)^p}{\sum_{w' \in C(v)} \pi(w')^p} \mu_k(v),$$

for all $v \in \mathcal{S}_k$ and $w \in C(v)$, where $C(v)$ is as defined in (2.2).

By (H4), (H1) and the fact that every vertex has at least one child, we obtain

$$\eta_-^p \pi(v)^p \leq \sum_{w' \in C(v)} \pi(w')^p \leq \pi(v)^p.$$

Therefore, we have

$$\frac{\mu_k(v)}{\pi(v)^p} \leq \frac{f_0(w)}{\pi(w)^p} \leq \eta_-^{-p} \frac{\mu_k(v)}{\pi(v)^p} \leq C \frac{f_0(w)}{\pi(w)^p} \tag{3.25}$$

for all points $v \in \mathcal{S}_k$ and $w \in C(v)$. Note that every point $v \in \mathcal{S}_k$ has at least one child, because we always have $(\pi_1(v), \pi_2(v) + 1) \in C(v)$ for any $v \in \mathcal{S}$. This implies f_0 is probability mass function on $\mathcal{S}_{k+1}$ such that (μ_k, f_0) is (C, π)-compatible as shown in (3.25).

Let f_T denote the probability mass function constructed from μ_k and f_0 as given by Lemma 3.14. We claim that $\mu_{k+1} := f_T$ is the probability mass function on $\mathcal{S}_{k+1}$ with the desired properties. Next, we show that μ_{k+1} satisfies the conditions.

(1) This is an immediate consequence of Lemma 3.14(a).

(2) This follows from Lemma 3.14(b).

(3) It remains to verify condition (3). Since $d(\pi_1(v), \pi_1(w)) < a^{-k}$ for all $w \in C(v), v \in \mathcal{S}_k$, there was a mass transfer over a distance of at most a^{-k} while passing from μ_k to f_0. Therefore it suffices to verify that while passing from f_0 to $f_T = \mu_{k+1}$ there is a transfer over a distance of at most $2\lambda a^{-k-1}$. Since $d(\pi_1(w), \pi_1(w')) < 2\lambda a^{-k-1}$ for all points $w, w' \in \mathcal{S}_{k+1}$ such that $D_2(w, w') = 1$, the desired conclusion follows from Lemma 3.14(c).

□

We construct a doubling measure on (X, d) in Lemma 3.15 using Lemma 3.11.

Lemma 3.15 (Construction of doubling measure)**.** *Let (X, d) be a compact doubling metric space and let $\mathcal{S}$ denote a hyperbolic filling with vertical and horizontal parameters*

a, λ respectively with $a \geq \lambda \geq 6$. Let $\rho : \mathcal{S} \to (0,\infty)$ denote a weight function that satisfies the hypotheses (H1), (H2), and (H4). Let μ_0 denote the (unique) probability measure on $\mathcal{S}_0 = \{v_0\}$. Let μ_k denote the probability measure on $\mathcal{S}_k$ for all $k \in \mathbb{N}$ constructed inductively using Lemma 3.11. Let

$$\widetilde{\mu}_k = \sum_{v \in \mathcal{S}_k} \mu_k(v) \delta_{\pi_1(v)} \quad \textit{for all } k \in \mathbb{N}_{\geq 0},$$

denote a sequence of probability measures on X associated with the above construction. Then any sub-sequential weak limit μ of $(\widetilde{\mu}_k)_{k\in\mathbb{N}}$ is a doubling measure on (X,d).

Proof. Observe that such a sub-sequential limit μ exists by Prokhorov's theorem along with the compactness of (X,d).

Since $\operatorname{diam}(X,d) = \frac{1}{2}$, it suffices to consider $r < 1$. For $x \in X$ choose a sequence $\{v_n\}$ as given in Lemma 3.8. We obtain two sided bounds on $\mu(B(x,r))$ using $\mu_n(v_n)$ for a suitably chosen value of n. To describe this let $n \in \mathbb{N}_{\geq 0}$ denote the largest integer such that $a^{-n} \geq r$. We claim that

$$\mu(B(x,r)) \asymp \mu_n(v_n) \tag{3.26}$$

where the constants of comparison are independent of $x \in X$, $r \in (0,1)$. Let us first show the upper bound. If mass from $\mu_n(v), v \in \mathcal{S}_n$ contributes to $\mu(\overline{B(x,r)})$, then by Lemma 3.11(3) we have

$$d(\pi_1(v), x) \leq r + \sum_{k=n}^{\infty} (1+2\lambda a^{-1}) a^{-k} = \left(1 + (1+2\lambda a^{-1})(1-a^{-1})^{-1}\right) a^{-n}$$

Since $\lambda > ((1-a^{-1})^{-1})(1+2\lambda a^{-1})$, we have that $x \in B(v, \lambda a^{-n}) \cap B(v_n, \lambda a^{-n})$ and hence $D_2(v, v_n) \leq 1$. Therefore

$$\mu(B(x,r)) \leq \sum_{v \in \mathcal{S}_n : D_2(v,v_n) \leq 1} \mu_n(v).$$

If $D_2(v,v_n) \leq 1$ and $v \in \mathcal{S}_n$, then by Lemma 3.11(1) and (H2), we obtain $\mu_n(v_n) \asymp \mu(v)$ for any pair of such vertices. Furthermore since (X,d) satisfies the metric doubling property, the number of neighbors of each vertex is uniformly bounded above [BBS, Proposition 4.5]. Combining the above estimates yields the upper bound in (3.26).

For the lower bound, we consider $\mu_{n+2}(v_{n+2})$. By Lemma 3.11(3) and $d(\pi_1(v_{n+2}), x) < (1+2\lambda a^{-1}) a^{-(n+2)}$, we note that the mass from v_{n+2} stays within $B(x,s)$ where

$$s = (1+2\lambda a^{-1}) \left(1 + (1-a^{-1})^{-1}\right) a^{-(n+2)} < r$$

(since $a^{-n-1} < r$ and $a^{-1}(1+2\lambda a^{-1})(1-a^{-1})^{-1} < 1$). This implies that $\mu(B(x,r)) \geq \mu_{n+2}(v_{n+2})$. This along with Lemma 3.11(2) and (H1), we obtain $\mu_{n+2}(v_{n+2}) \asymp \mu_n(v_n)$. Combining these estimates yields the lower bound for $\mu(B(x,r))$ in (3.26).

Next, we show that (3.26) implies the desired doubling property. For the remainder of the proof we assume $r \in (0, 1/2)$. The case $r \geq 1/2$ is similar and easier. Let $N \in \mathbb{N}_{\geq 0}$

denote th largest integer such that $a^{-N} \geq 2r$. This implies $a^{-(N+2)} < 2a^{-1}r < r$. This implies that $n = N$ or $n = N + 1$. Therefore by the same argument as above (using Lemma 3.11(2) and (H1)), we have $\mu_n(v_n) \asymp \mu_N(v_N)$. This along with (3.26) shows that μ is a doubling measure on (X, d). $\square$

Let Θ_ρ denote the metric defined in (3.8). In the following proposition, we obtain upper bound on the Assouad dimension of (X, Θ_ρ). We establish this by showing that the measure μ in Lemma 3.15 is p-homogeneous in (X, Θ_ρ). This along with Theorem 3.1 shows that $\dim_A(X, \Theta_\rho) \leq p$.

Proposition 3.16. *Let (X, d) be a compact doubling metric space and let $\mathcal{S}$ denote a hyperbolic filling with vertical and horizontal parameters a, λ respectively such that $a \geq \lambda \geq 6$. Let $\rho : \mathcal{S} \to (0, \infty)$ denote a weight function that satisfies the hypotheses (H1), (H2), (H3), and (H4). Let Θ_ρ denote the metric defined in (3.8) using Proposition 3.6. Then the measure μ defined in Lemma 3.15 is p-homogeneous in (X, Θ_ρ), where p is the constant in (H4). In particular, $\dim_{\mathrm{A}}(X, \Theta_\rho) \leq p$.*

Proof. For ease of notation, we abbreviate Θ_ρ by Θ. By (3.7), there exists $L > 1$ such that (3.9) holds. Let η_-, η_+, K_0 denote the constants in (H1) and (H2).

Next, we show that μ is p-homogeneous in (X, Θ); that is, there exists $C > 1$ such that

$$\frac{\mu(B_\Theta(x, r))}{\mu(B_\Theta(x, s))} \leq C \left(\frac{r}{s}\right)^p \quad \text{for all } x \in X, 0 < s < r. \tag{3.27}$$

Let $0 < s < r$ and $x \in X$. Choose a sequence $(v_n)_{n \in \mathbb{N}_{\geq 0}}$ such that $v_n \in \mathcal{S}_n$ for all $n \in \mathbb{N}_{\geq 0}$ as given in Lemma 3.8. Let $k \in \mathbb{N}_{\geq 0}$ be the smallest non-negative integer such that $\pi(v_k) < K_0^{-1}s$. By Lemma 3.9, Lemma 3.15 and (3.26), we have

$$\mu(B_\Theta(x, s)) \gtrsim \mu_k(v_k). \tag{3.28}$$

If $k = 0$, then it we have $1 \geq \mu(B_\Theta(x, r)) \geq \mu(B_\Theta(x, s)) \gtrsim 1$ which implies (3.27). So it suffices to consider the case $k \geq 1$. The choice of k along with (H1) implies that

$$\eta_- K_0^{-1}s \leq \eta_- \pi(v_{k-1}) \leq \pi(v_k) < K_0^{-1}s. \tag{3.29}$$

Next, we bound $\mu(B_\Theta(x, r))$ from above. We consider two cases depending on whether or not $r < (K_0 L)^{-1}\pi(v_0)$. If $r < (K_0 L)^{-1}\pi(v_0)$, there exists $l \in \mathbb{N}$ such that $\pi(v_l) \leq K_0 L r$ and $\pi(v_{l-1}) > K_0 L r$. Hence by (H1), we have

$$\eta_- K_0 L r < \eta_- \pi(v_{l-1}) \leq \pi(v_l) \leq K_0 L r. \tag{3.30}$$

By Lemma 3.9, Lemma 3.15, and (3.26), we have

$$\mu(B_\Theta(x, r)) \leq \mu\left(B_d(\pi_1(v_l), 2(\lambda + 2)a^{-l})\right) \lesssim \mu\left(B_d(\pi_1(v_l), a^{-l})\right) \lesssim \mu_l(v_l). \tag{3.31}$$

Since $r > s$, we have $l \leq k$. Therefore by Lemma 3.11(2), we have

$$\frac{\mu_l(v_l)}{\pi(v_l)^p} \leq \frac{\mu_k(v_k)}{\pi(v_k)^p}, \quad \text{for any } l \leq k. \tag{3.32}$$

By (3.28), (3.31), (3.29), (3.30) and (3.32), we have

$$\frac{\mu(B_\Theta(x,r))}{\mu(B_\Theta(x,s))} \lesssim \frac{\mu_l(v_l)}{\mu_k(v_k)} \lesssim \frac{\pi(v_l)^p}{\pi(v_k)^p} \asymp \frac{r^p}{s^p}.$$

This implies (3.27) in the case $r < (K_0 L)^{-1}\pi(v_0)$.

On the other hand, if $r \geq (K_0 L)^{-1}\pi(v_0)$ we use the trivial bound $\mu(B_\Theta(x,r)) \leq 1 = \mu_0(v_0)$. By (3.28), (3.29), (3.32) and the bound $1 \asymp \pi(v_0) \lesssim r$, we have

$$\frac{\mu(B_\Theta(x,r))}{\mu(B_\Theta(x,s))} \lesssim \frac{\mu_0(v_0)}{\mu_k(v_k)} \lesssim \frac{\pi(v_0)^p}{\pi(v_k)^p} \lesssim \frac{r^p}{s^p}.$$

This completes the proof of (3.27). By Theorem 3.1, we obtain the desired bound on Assouad dimension. □

Proof of Theorem 3.3. This follows immediately from Corollary 3.7 and Proposition 3.16. □

3.5 Upper bound on Assouad dimension using weights

In this subsection, we prove Theorem 3.4. The proof of the Theorem 3.4 is very similar to that of [Car, Theorem 1.2] except for the use of Theorem 3.3 instead of [Car, Theorem 1.1]. For the convenience of the reader, we provide further details since the hypothesis (H4) is different from $\widetilde{\text{(H4)}}$ of [Car]. To the reader who is familiar with Carrasco's work, we point out that the estimate in [Car, (2.52)] implies our version of (H4) for small enough η_0. The proof of other three hypothesis is similar. Readers who are familiar with the proof of [Car, Theorem 1.2] may want to skip the proof of Theorem 3.4.

Let $\rho : \mathcal{S} \to [0,\infty)$ be a function. We define $\rho^* : \mathcal{S} \to [0,\infty)$ as

$$\rho^*(v) = \min\{\rho(w) : w \in \mathcal{S} : \pi_2(w) = \pi_2(v), D_2(v,w) \leq 1\} \quad \text{for all } v \in \mathcal{S}. \tag{3.33}$$

Similarly, we define $\pi^* : \mathcal{S} \to [0,\infty)$ as

$$\pi^*(v) = \min\{\pi(w) : w \in \mathcal{S} : \pi_2(w) = \pi_2(v), D_2(v,w) \leq 1\} \quad \text{for all } v \in \mathcal{S}. \tag{3.34}$$

If $\gamma = (v_1, \ldots, v_N)$ is a horizontal path, we define

$$L_h(\gamma,\rho) = \sum_{j=1}^{N-1} \rho^*(v_j) \wedge \rho^*(v_{j+1}). \tag{3.35}$$

We introduce the following hypothesis on $\rho : \mathcal{S} \to [0,\infty)$ which serves as a simpler sufficient condition for (H3):

(H3') for all $k \geq 1$, for all $v \in \mathcal{S}_k$ and for all $\gamma \in \Gamma_{k+1}(v)$, it holds $L_h(\gamma,\rho) \geq 1$, where $L_h(\gamma,\rho)$ is as defined in (3.35). The hypothesis (H3') is simpler to verify than (H3) because it only involves curves with horizontal edges. The following is a version of [Car, Proposition 2.9] and provides a useful sufficient condition for (H3).

Proposition 3.17. *Let (X,d) be a compact doubling metric space. Let $(\mathcal{S},D_2)$ denote the hyperbolic filling with horizontal and vertial parameters λ, a respectively that satisfy $a \ge \lambda \ge 6$. Assume that there exists $p>0$ and a function $\rho : \mathcal{S} \to (0,\infty)$ which satisfy the hypotheses (H1), (H2), and (H3'). Then the function ρ also satisfies (H3).*

The proof of Proposition 3.17 requires several lemmas. We say that a path $\gamma = (v_1,\ldots,v_N)$ is of *level k* (resp. *level at most k*) if $\pi_2(v_i)=k$ (resp. $\pi_2(v_i)\le k$) for all $i=1,\ldots,N$.

Lemma 3.18. *(Cf. [Car, Lemma 2.10]) Let (X,d) and $(\mathcal{S},D_2)$ be as given in Proposition 3.17. Let $k\ge 0$ and $v\in\mathcal{S}_k$. Assume that ρ satisfies (H3'). Consider a horizontal path $\gamma=(v_1,\ldots,v_n)$ of level $k+1$ such that $\pi_1(v_i)\in B(\pi_1(v),3a^{-k})$ for all $i=1,\ldots,N$, $\pi_1(v_1)\in B(\pi_1(v),a^{-k})$ and $\pi_1(v_N)\notin B(\pi_1(v),2a^{-k})$. Let w denote the parent of z_1. Then*

$$\sum_{i=1}^{N-1}\pi^*(v_i)\wedge\pi^*(v_{i+1}) \ge \max\{\pi^*(v),\pi^*(w)\}.$$

Proof. First, we show that for all $j=1,\ldots,N$,

$$\pi^*(v_j) \ge \max\{\pi^*(v),\pi^*(w)\}\min\{\rho(w_j): w_j\in\mathcal{S}_{k+1}, D_2(w_j,v_j)\le 1\}. \tag{3.36}$$

Let $\widetilde{w}_j\in\mathcal{S}_{k+1}$ be such that $\pi^*(v_j)=\pi(\widetilde{w}_j)$ and $D_2(\widetilde{w}_j,v_j)\le 1$. Let $u_j\in\mathcal{S}_k$ be the parent of w_j. Then by Lemma 2.4(a),

$$\begin{aligned} d(\pi_1(v),\pi_1(u_j)) &\le d(\pi_1(v),\pi_1(v_j)) + d(\pi_1(v_j),\pi_1(\widetilde{w}_j)) + d(\pi_1(\widetilde{w}_j),\pi_1(u_j)) \\ &< 3a^{-k} + 2\lambda a^{-k-1} + a^{-k} = (4+2\lambda a^{-1})a^{-k} < \lambda a^{-k}, \\ d(\pi_1(w),\pi_1(v)) &\le d(\pi_1(w),\pi_1(v_1)) + d(\pi_1(v_1),\pi_1(v)) < a^{-k}+a^{-k} < \lambda a^{-k} \end{aligned}$$

The above estimates imply that $D_2(v,u_j)\le 1$ and $D_2(w,u_j)\le 1$. Therefore $\pi(u_j)\ge \max\{\pi^*(v),\pi^*(w)\}$ and hence

$$\begin{aligned} \pi^*(v_j) &= \pi(\widetilde{w}_j) = \pi(u_j)\rho(\widetilde{w}_j) \\ &\ge \max\{\pi^*(v),\pi^*(w)\}\min\{\rho(w_j): w_j\in\mathcal{S}_{k+1}, D_2(w_j,v_j)\le 1\}. \end{aligned}$$

This completes the proof of (3.36). Therefore, we have

$$\begin{aligned} \sum_{i=1}^{N-1}\pi^*(v_i)\wedge\pi^*(v_{i+1}) &\overset{(3.36)}{\ge} \max\{\pi^*(v),\pi^*(w)\}\sum_{i=1}^{N-1}\rho^*(v_i)\wedge\rho^*(v_{i+1}) \\ &= \max\{\pi^*(v),\pi^*(w)\}L_h(\gamma,\rho) \ge \max\{\pi^*(v),\pi^*(w)\}, \end{aligned}$$

where we use (3.36) in the first line and (H3') and the second line above. $\square$

We introduce a different notion of length on paths. For any edge $e = \{u, v\}$ we define

$$\widehat{\ell}_1(e) = \begin{cases} \pi^*(u) \wedge \pi^*(v) & \text{if } e = \{u, v\} \text{ is a horizontal edge,} \\ K_0 \eta_-^{-1} \pi^*(v) & \text{if } e = \{u, v\} \text{ and } u \text{ is a parent of } v, \end{cases} \tag{3.37}$$

and for a path $\gamma = (v_1, \ldots, v_N)$, we define

$$\widehat{\ell}_1(\gamma) = \sum_{i=1}^{N-1} \widehat{\ell}_1(e_i), \quad \text{where } e_i = \{v_i, v_{i+1}\}. \tag{3.38}$$

If $w \in \mathcal{S}_{k+1}, u, v \in \mathcal{S}_k$ such that $D_2(u, v) = D_2(u, w) = 1$, then by (H1) and (H2), we have

$$\widehat{\ell}_1(\{u, v\}) \le \pi^*(u) \le \pi(u) \le \eta_-^{-1} \pi(w) \le K_0 \eta_-^{-1} \pi^*(w) \le \widehat{\ell}_1(\{u, w\}). \tag{3.39}$$

Lemma 3.19. *[Cf. [Car, Lemma 2.11]] Let (X, d) and $(\mathcal{S}, D_2)$ be as given in Proposition 3.17. Assume that $\rho : \mathcal{S} \to [0, \infty)$ satisfies the hypotheses (H1), (H2), (H3'). Let $u, v \in \mathcal{S}_{k+1}$ be such that $d(\pi_1(u), \pi_1(v)) > 4a^{-k}$. Let $\gamma = (v_1, \ldots, v_N)$ be a path of level at most $k+1$ from $v_1 = u$ to $v_N = v$. Then there exists a path $\gamma' = (u_1, \ldots, u_M)$ of level at most k such that:*

1. *u_1, u_M are parents of v_1 and v_N respectively, and*
2. *$\widehat{\ell}_1(\gamma') \le \widehat{\ell}_1(\gamma)$.*

Proof. Let $\gamma = (v_1, \ldots, v_N)$ be a path of level at most $k+1$ as given in the statement of the lemma. We decompose γ into sub-paths of level at most k or level equal to $k+1$. Let $s_1 = 1$. Define inductively positive integers s_i, t_i as

$$t_i = \min\{j > s_i : \pi_2(v_j) \le k \text{ or } j = N\},$$
$$s_{i+1} = \min\{j \ge t_i : \pi_2(v_{j+1}) = k+1\}.$$

We stop when $t_i = N$ for some $i = L$. Note that $\pi_2(v_{s_1}) = \pi_2(v_{t_L}) = k+1$, and $\pi_2(v_{s_i}) = \pi_2(v_{t_j}) = k$ for $i \ne 1$ and $j \ne L$. Since we are trying to bound $\widehat{\ell}_1(\gamma)$ from below, we may assume that path γ has no self-intersections; that is $v_i \ne v_j$ for all $i \ne j$. In particular, $v_{s_i} \ne v_{t_i}$ for all i.

For each $i = 1, \ldots, L$, let γ_i denote the sub-path $(v_{s_i}, \ldots, v_{t_i - 1})$. We will replace each path γ_i with γ_i' such that $\widehat{\ell}_1(\gamma_i') \le \widehat{\ell}_1(\gamma_i)$.

First, we consider $2 \le i \le L-1$ and postpone the cases $i = 1, L$ to the end. Let $2 \le i \le L-1$. We consider two cases.

Case 1: $\pi_1(v_j) \in B(\pi_1(v_{s_i}), 2a^{-k})$ for all $j = s_i + 1, \ldots, t_i - 1$.

In this case, $v_{s_i}, v_{t_i} \in \mathcal{S}_k$ and are the parents of v_{s_i+1}, v_{t_i-1} respectively. By Lemma 2.4(c), $D_2(v_{s_i}, v_{t_i}) = 1$ and hence we replace γ_i with $\gamma_i' = (v_{s_i}, v_{t_i})$. From (3.39), we obtain

$$\widehat{\ell}_1(\gamma_i') \le \widehat{\ell}_1(\{v_{s_i}, v_{s_i+1}\}) \le \widehat{\ell}_1(\gamma_i).$$

Case 2: There exists $j_1 \in \{s_i+1, t_i-1\}$ such that $\pi_1(v_{j_1}) \notin B(\pi_1(v_{s_1}, 2a^{-k}))$. We assume j_1 is the first index with this property. We denote $j_0 = s_i + 1, w_0 = v_{s_i} \in \mathcal{S}_k$. Suppose j_l, w_l are defined, and if $j_l < t_i - 1$, we define

$$j_{l+1} = \min\{j_l < j < t_i - 1 : \pi_1(v_j) \notin B(\pi_1(w_l), 2a^{-k}) \text{ or } j = t_i - 1\},$$

and let $w_{l+1} \in \mathcal{S}_k$ be the parent of $v_{j_l+1} \in \mathcal{S}_{k+1}$. Let L_i be such that $j_{L_i} = t_i - 1$.

If $l \in \{0, \ldots, L_i - 2\}$, we have $\pi_1(v_{j_{l+1}}) \notin B(\pi_1(w_l), 2a^{-k})$. Since $a > 2\lambda$, we have $d(\pi_1(w_l), \pi_1(v_{j_l+1})) \leq d(\pi_1(w_l), \pi_1(v_{j_l})) + d(\pi_1(v_{j_l}), \pi_1(v_{j_l+1})) < 2a^{-k} + 2\lambda a^{-k-1} < 3a^{-k}$. Therefore by Lemma 3.18, Lemma 2.4(c), and (3.39), we have

$$\widehat{\ell}_1((w_l, w_{l+1})) \leq \pi^*(w_l) \leq \widehat{\ell}_1\left((v_{j_l}, \ldots, v_{j_{l+1}-1})\right) \quad \text{for all } l = 0, \ldots, L_i - 2. \tag{3.40}$$

The above estimate (3.40) is also true for $j = L_i - 1$ by combining the above argument and with that of case 1 by considering depending on whether or not $\pi_1(v_{j_1}) \notin B(\pi_1(v_{s_1}, 2a^{-k}))$. Hence $\gamma_i' = (w_0, \ldots, w_{L_i-1}, w_{L_i})$, where $w_0 = v_{s_i}, w_{L_i} = v_{t_i}$. By (3.40) along with the above remark, we obtain

$$\widehat{\ell}_1(\gamma_i') \leq \widehat{\ell}_1(\gamma_i), \quad \text{for } i \in \{2, \ldots, L-1\}.$$

The case $i = 1$ is also similar to above. Let u_1 be the parent of v_1. Similar to argument above, we consider two cases depending on whether or not $\pi_1(v_j) \in B(\pi_1(u_1), 2a^{-k})$ for all $j = 1, \ldots, t_i - 1$ as explained in [Car, proof of Lemma 2.11]. This yields a path γ_1' from u_1 to v_{t_1}. The case $i = L$ is exactly same as $i = 1$ after reversing the order in which the vertices of γ_L appear. By concatenating the paths $\gamma_1', \ldots, \gamma_L'$, we obtain the path $(u_1, \ldots, u_M)$ with desired properties. ☐

Lemma 3.20. *(Cf. [Car, Lemma 2.12]) Let (X, d) and $(\mathcal{S}, D_2)$ be as given in Proposition 3.17. Assume that $\rho : \mathcal{S} \to [0, \infty)$ satisfies the hypotheses (H1), (H2), and (H3'). There exists a constant $K_2 \geq 1$ such that the following property: for all $x, y \in X$, there exists k_0 depending on x, y such that for all $k \geq k_0$, if $u, v \in \mathcal{S}_k$ such that $x \in B_u, y \in B_v$, then any path γ joining u and v satisfies*

$$\widehat{\ell}_1(\gamma) \geq K_2^{-1} \pi(c(x, y)).$$

Proof. Let $u, v \in \mathcal{S}_k$ be such that $x \in B_u, y \in B_v$. Let m be such that $\pi_2(w) = m$ for some (or equivalently, for all) $w \in c(x, y)$. By (2.8), we have $d(x, y) > a^{-m-1}$. For $k \geq m + 2$, we have (using $a \geq 12$)

$$d(\pi_1(u), \pi_1(v)) \geq d(x, y) - d(x, \pi_1(u)) - d(y, \pi_1(v)) > a^{-m-1} - 2a^{-m-2} \geq 10a^{-m-2}. \tag{3.41}$$

The idea is to use Lemma 3.18 to find a path of level at most $m + 2$ whose $\widehat{\ell}_1$ length is larger than $\widehat{\ell}_1(\gamma)$. We consider two cases.

Case 1: The path γ is of level at most k, where $k \geq m + 2$. By (3.41), we can apply Lemma 3.18. Set $\gamma_k = \gamma$. Let $u_l, v_l \in \mathcal{S}_l$ be such that u, v are descendants of u_l, v_l respectively. By Lemma 2.4(a), for all $l \geq m + 2$, we have

$$\begin{aligned} d(\pi_1(u_l), \pi_1(v_l)) &\geq d(\pi_1(u), \pi_1(v)) - d(\pi_1(u), \pi_1(u_l)) - d(\pi_1(v), \pi_1(v_l)) \\ &\geq 10a^{-m-2} - 2\frac{a}{a-1}a^{-l} > 6a^{-m+2}. \end{aligned}$$

Using the above estimate, and applying Lemma 3.18 repeatedly we obtain path γ_{m+2} of level at most $m+2$ from u_{m+2} to v_{m+2} such that $\widehat{\ell}_1(\gamma) \geq \widehat{\ell}_1(\gamma_{m+2})$. This along with (3.39), (H1), (H2), implies

$$\widehat{\ell}_1(\gamma) \geq K_0^{-2}\pi(u_{m+2}) \geq K_0^{-2}\eta_-^2\pi(u_m) \geq K_0^{-3}\eta_-^2\pi(c(x,y)).$$

In the last estimate, we used $D_2(u_m, w) \leq 1$ for any $w \in c(x,y)$ (since $x \in B(\pi_1(u_m), \lambda a^{-m}) \cap B(\pi_1(w), \lambda a^{-m}) \neq \emptyset$).

Case 2: γ is not a path of level at most k. Let $n > k$ be the smallest integer such that γ is a path of level at most n. Let $k_0 \geq m+2$ be large enough so that

$$K_0^{-3}\eta_-^2\eta_-^m \geq 4K_0\eta_-^{-1}\sum_{i=k_0}^{\infty}\eta_+^i.$$

Let $\widetilde{u}_n, \widetilde{v}_n \in \mathcal{S}_n$ be such that $x \in B_{\widetilde{u}_n}, y \in B_{\widetilde{v}_n}$ and let $\widetilde{u}_k, \widetilde{v}_k \in \mathcal{S}_k$ be the ancestors of $\widetilde{u}_n, \widetilde{v}_n$ respectively. By Lemma 2.4(a), $D_2(\widetilde{u}_n, u_n) \leq 1$ and $D_2(\widetilde{v}_n, v_n) \leq 1$. Let γ_u denote the path from $\widetilde{u}_n$ to u formed by concatenating the genealogy from $\widetilde{u}_n$ to $\widetilde{u}_k$ and adding an edge from $\widetilde{u}_k$ to u_k if necessary. Similarly, let γ_v denote the path from v to $\widetilde{v}_n$ formed in a similar fashion. By concatenating $\gamma_u, \gamma, \gamma_v$ we obtain a path $\widetilde{\gamma}$ from $\widetilde{v}_n$ to $\widetilde{u}_n$ whose level is at most n. Using the first case, we obtain

$$\ell_1(\widetilde{\gamma}) \geq K_0^{-3}\eta_-^2\pi(c(x,y)) \geq K_0^{-3}\eta_-^2\eta_-^m \geq 4K_0\eta_-^{-1}\sum_{i=k_0}^{\infty}\eta_+^i \geq 2\ell_1(\gamma_u) + 2\ell_1(\gamma_v).$$

This implies

$$\ell_1(\gamma) \geq \frac{1}{2}K_0^{-3}\eta_-^2\pi(c(x,y))$$

for any $k \geq k_0$. $\square$

Proof of Proposition 3.17. By (H1), (H2), there exists $c > 0$ such that

$$L_\rho(\gamma) \geq c\widehat{\ell}_1(\gamma) \quad \text{for all paths } \gamma \text{ in } (\mathcal{S}, D_2).$$

This estimate along with Lemma 3.20 implies (H3). $\square$

The statement of the lemma below is slightly different from that of [Car, Lemma 2.13] and the proof is omitted as it is similar to [Car].

Lemma 3.21. *([Car, Lemma 2.13]) Let (X,d) and $(\mathcal{S}, D_2)$ be as given in Theorem 3.4. Suppose we have a function $\pi_0 : \mathcal{S}_k \to (0,\infty)$ such that*

$$\frac{1}{K} \leq \frac{\pi_0(v)}{\pi_0(w)} \leq K \quad \textit{for all } v, w \in \mathcal{S}_k \textit{ such that } D_2(v,w) \leq 1, \tag{3.42}$$

where $K \geq 1$ is a constant. Suppose also that there is function $\pi_1 : \mathcal{S}_{k+1} \to (0,\infty)$ such that for any $u \in \mathcal{S}_k$ and for any $v \in \mathcal{S}_{k+1}$ such that u is the parent of v, we have

$$1 \leq \frac{\pi_0(u)}{\pi_1(v)} \leq K. \tag{3.43}$$

Let $\widehat{\pi}_1 : \mathcal{S}_{k+1} \to [0,\infty)$ be defined as

$$\widehat{\pi}_1(w) = \pi_1(w) \vee \left(\frac{1}{K} \max\{\pi_1(v) : v \in \mathcal{S}_{k+1}, D_2(v,w) \le 1\} \right). \tag{3.44}$$

Then for all $w_1, w_2 \in \mathcal{S}_{k+1}$ such that $D_2(w_1,w_2) \le 1$, we have

$$\frac{1}{K} \le \frac{\widehat{\pi}_1(w_1)}{\widehat{\pi}_1(w_2)} \le K. \tag{3.45}$$

Lemma 3.22. *(see [Car, Lemma 2.14]) Let $G = (V,E)$ be a graph whose vertices has a degree bounded by K and let $p > 0$. Let Γ be a family of paths of G. Let $\tau : V \to [0,\infty)$ that satisfies*

$$\sum_{i=1}^{N-1} \tau(v_i) \ge 1 \quad \textit{for all paths } \gamma = (v_1, \ldots, v_N) \in \Gamma.$$

Let $d_G : V \times V \to [0,\infty)$ denote the combinatorial graph distance metric on V. Let $\widehat{\tau} : V \to [0,\infty)$ be defined as

$$\widehat{\tau}(v) = 2 \max\{\tau(w) : w \in V, d_G(w,v) \le 2\}.$$

Then

$$\sum_{i=1}^{N-1} \widehat{\tau}^*(v_i) \wedge \widehat{\tau}^*(v_{i+1}) \ge 1 \quad \textit{for all paths } \gamma = (v_1, \ldots, v_N) \in \Gamma,$$

where $\widehat{\tau}^*(v) = \min\{\widehat{\tau}(w) : d_G(w,v) \le 1\}$, and such that

$$\sum_{v \in V} \widehat{\tau}(v)^p \le 2^p (K^2 + 1) \sum_{v \in V} \tau(v)^p.$$

The statement of Lemma 3.22 is slightly different from that of [Car, Lemma 2.14] where the term $K^2 + 1$ was replaced by K^2. This is because the estimate $\#\{w \in V : d_G(w,v) \le 2\} \le K^2$ for all $v \in V$ in [Car] must be replaced by $\#\{w \in V : d_G(w,v) \le 2\} \le K^2 + 1$. The proof is otherwise identical and is omitted.

Proof of Theorem 3.4. Let $\eta_0 \in (0,1)$ whose value will be determined later. Since (X,d) is doubling there exists $M_1 \in \mathbb{N}$, depending only on a, λ and the doubling constant such that the number of neighbors of each vertex in $(\mathcal{S}, D_1)$ is bounded by M_1, and in particular the number of children of each vertex uniformly bounded above [BBS, Proposition 4.5]. Set

$$\eta_- = \left(\eta_0 M_1^{-1}\right)^{1/p} \in (0,1).$$

Let $\sigma : \mathcal{S} \to [0,\infty)$ satisfy (S1) and (S2). We define $\tau : \mathcal{S} \to [0,\infty)$ as $\tau := (\sigma^p + \eta_-^p)^{1/p} \ge \eta_-$, which also satisfies (S1). The function τ satisfies

$$\sum_{v \in C(u)} \tau(v)^p \le \sum_{v \in C(u)} (\sigma(v)^p + \eta_-^p) \le 2\eta_0 \quad \text{for all } u \in \mathcal{S}. \tag{3.46}$$

By [Hei, Exercise 10.17], there exists M_2, depending only on λ and doubling constant of (X,d), such that

$$\#\{w \in \mathcal{S} : D_2(v,w) = 1, \pi_2(w) = \pi_2(v)\} \le M_2 \quad \text{for all } v \in \mathcal{S}. \tag{3.47}$$

That is, the number of horizontal edges at any vertex is uniformly bounded in M_2. By Lemma 3.22, the function

$$\widehat{\tau}(v) := 2\max\{\tau(w) : w \in \mathcal{S}_{\pi_2(v)}, D_2(v,w) \le 2\}$$

satisfies condition (H3') and

$$\begin{aligned}
\sum_{v \in C(u)} \widehat{\tau}(v)^p &\le 2^p(M_2^2+1) \sum_{\substack{w \in \mathcal{S}_{\pi_2(u)+1},\\ D_2(w,v)\le 2}} \tau(w)^p \quad \text{(by Lemma 3.22)}\\
&\le 2^p(M_2^2+1) \sum_{\substack{\widetilde{u} \in \mathcal{S}_{\pi_2(u)},\\ D_2(w,v)\le 1}} \sum_{w \in C(\widetilde{u})} \tau(w)^p \quad \text{(by Lemma 2.4(b))}\\
&\le 2^{p+1}(M_2^2+1) \sum_{\substack{\widetilde{u} \in \mathcal{S}_{\pi_2(u)},\\ D_2(w,v)\le 1}} \eta_0 \quad \text{(by (S2))}\\
&\le 2^{p+1}(M_2^2+1)(M_2+1)\eta_0 \quad \text{(by (3.47))}.
\end{aligned} \tag{3.48}$$

We construct a function $\rho : \mathcal{S} \to [0,\infty)$ that satisfies

(1) $\rho(v) \ge \widehat{\tau}(v)$ for all $v \in \mathcal{S}$.

(2) ρ satisfies (H2) with $K_0 = \eta_-^{-1}$.

(3) $\rho(v) \le \max\{\widehat{\tau}(w) : D_2(w,v) \le 1, \pi_2(w) = \pi_2(v)\}$ for all $v \in \mathcal{S}$.

The idea behind the proof is to inductively construct ρ on $\mathcal{S}_k$ for $k = 0,1,\ldots$. Since the conditions (2) and (3) depend only on horizontal edges this inductive construction works well. We pick $\rho(v_0) = \widehat{\tau}(v_0)$ where $v_0 \in \mathcal{S}_0$. Clearly, this satisfies (1), (2), (3) on $\mathcal{S}_0$ because $\mathcal{S}_0$ is a singleton set. Suppose we have constructed ρ on $\cup_{j=0}^{i}\mathcal{S}_j$, we construct ρ on $\mathcal{S}_{i+1}$ as follows. Define $\pi_0 : \mathcal{S}_i \to (0,\infty), \pi_1 : \mathcal{S}_{i+1} \to (0,\infty)$ as

$$\pi_0(u) = \prod_{w \in g(u)} \rho(w), \quad \pi_1(v) = \widehat{\tau}(v) \prod_{w \in g(v), w \ne v} \rho(w) = \widehat{\tau}(v)\pi_0(\widetilde{v})$$

for all $u \in \mathcal{S}_i, v \in \mathcal{S}_{i+1}$, where $\widetilde{v} \in \mathcal{S}_i$ is the parent of $v \in \mathcal{S}_{i+1}$. Using the estimate $\widehat{\tau} \ge \eta_-$ along with induction hypothesis, π_0, π_i satisfy the hypotheses of Lemma 3.21 with $K = \eta_-^{-1}$. Consider the function $\widehat{\pi}_1 : \mathcal{S}_{i+1} \to (0,\infty)$ defined by (3.44) as

$$\widehat{\pi}_1(w) = \pi_1(w) \vee \left(\frac{1}{K}\max\{\pi_1(v) : v \in \mathcal{S}_{i+1}, D_2(v,w) \le 1\}\right),$$

and set $\rho : \mathcal{S}_{i+1} \to (0,\infty)$ as

$$\rho(w) = \frac{\widehat{\pi}_1(w)}{\pi_0(\widetilde{w})} \quad \text{for all } w \in \mathcal{S}_{i+1}, \text{ where } \widetilde{w} \text{ is the parent of } w.$$

Since $\widehat{\pi}_1 \geq \pi_1$ the condition (1) is satisfied. By Lemma 3.21, the condition (2) above is also satisfied on $\cup_{j=0}^{i+1}\mathcal{S}_j$. It only remains to check (3) on $\mathcal{S}_{i+1}$. For $v \in \mathcal{S}_{i+1}$, we have two possibilities for $\widehat{\pi}_1(v)$; either $\widehat{\pi}_1(v) = \pi_1(v)$ or $\widehat{\pi}_1(v) = K^{-1}\pi_1(w)$ for some $w \in \mathcal{S}_{i+1}$ such that $D_2(v,w) = 1$. The first possibility implies that $\rho(v) = \widehat{\tau}(v)$ and hence (3) is satisfied for v. The other possibility is that $\widehat{\pi}_1(v) = K^{-1}\pi_1(w) \geq \pi_1(v)$. In this case, let $\widetilde{v}, \widetilde{w} \in \mathcal{S}_i$ denote the parents of v, w respectively. By Lemma 2.4(b), $D_2(\widetilde{v},\widetilde{w}) \leq 1$. Therefore by condition (2) in the induction hypothesis, we have

$$\rho(v) = \frac{\widehat{\tau}(w)\pi_0(\widetilde{w})}{K\pi_0(\widetilde{v})} \leq \widehat{\tau}(w),$$

which concludes the proof of condition (3) above. By induction, there exists a function $\rho : \mathcal{S} \to (0,\infty)$ which satisfies (1), (2), (3) above.

Next, we want to show that ρ satisfies the upper bound $\rho \leq \eta_+$ in (H1) for some $\eta_+ \in (0,1)$ and the hypothesis (H4) whenever η_0 is small enough. To this end, consider

$$\begin{aligned}
\sum_{v \in C(u)} \rho(v)^p &\leq \sum_{v \in C(u)} \sum_{\substack{w \in \mathcal{S}: D_2(v,w) \leq 1, \\ \pi_2(v) = \pi_2(w)}} \widehat{\tau}(w)^p \quad \text{(by condition (3))} \\
&\leq (M_2+1) \sum_{\substack{\widetilde{u} \in \mathcal{S}: D_2(u,\widetilde{u}) \leq 1, \\ \pi_2(\widetilde{u}) = \pi_2(u)}} \sum_{v \in C(\widetilde{u})} \widehat{\tau}(v)^p \text{ (by (3.47) and Lemma 2.4(c))} \\
&\leq 2^{p+1}(M_2^2+1)(M_2+1)^3\eta_0 \quad \text{(by (3.47) and (3.48))}.
\end{aligned}$$

By the above estimate, the choice $\eta_0 \in (0,1)$ such that $2^{p+1}(M_2^2+1)(M_2+1)^3\eta_0 = 2^{-p}$ implies the upper bound $\rho \leq \eta_+$ in (H1) for $\eta_+ = \frac{1}{2} \in (0,1)$ and also (H4). Since $\widehat{\tau}$ satisfies (H3') and $\rho \geq \widehat{\tau}$, ρ also satisfies (H3'). This along with conditions (1), (2) above and Proposition 3.17 implies that ρ satisfies (H1), (H2), (H3), and (H4). The desired conclusion follows from Theorem 3.3. □

4 Critical exponent associated to the combinatorial modulus

Let $G = (V,E)$ be a graph and let Γ be a family of paths in G. Consider a function $\rho : V \to [0,\infty)$ and for $\gamma \in \Gamma$, we define its ρ-length as

$$\ell_\rho(\gamma) := \sum_{v \in \gamma} \rho(v),$$

and its p-mass by

$$M_p(\rho) = \sum_{v \in V} \rho(v)^p.$$

The p-combinatorial modulus[2] of Γ is defined as

$$\mathrm{Mod}_p(\Gamma, G) = \inf_{\rho \in \mathrm{Adm}(\Gamma)} M_p(\rho),$$

where $\mathrm{Adm}(\Gamma) := \{\rho : V \to [0, \infty) \mid \ell_\rho(\gamma) \geq 1 \text{ for all } \gamma \in \Gamma\}$ denote the set of Γ-admissible functions. If $\Gamma = \emptyset$, we set $\mathrm{Mod}_p(\Gamma, G) = 0$ by convention.

We recall the definition of critical exponent of the combinatorial modulus associated to a compact metric space (X, d). The idea behind the following definition is to approximate the compact metric space by a sequence of graphs G_k. Then the behavior of the modulus of (discrete) family of curves on G_k which 'cross an annulus' as $k \to \infty$ determines a critical exponent.

Definition 4.1. Let $a, \lambda, L \in (1, \infty)$ and $p > 0$ and let (X, d) be a compact metric space. Let X_k denote a maximal a^{-k} separated subset of X for all $k \geq 0$ and let $\mathcal{S}_k = \{(x, k) : x \in X_k\}$. In this section, we need not assume that X_k is increasing in k. Let $\pi_1 : \mathcal{S}_k \to X, \pi_2 : \mathcal{S}_k \to \mathbb{N}_{\geq 0}$ be the projection maps to the first and second components. For each $k \geq 1$, we define a graph G_k whose vertex set is $\mathcal{S}_k$ and there is an edge between distinct vertices v and w if and only if $B(\pi_1(v), \lambda a^{-\pi_2(v)}) \cap B(\pi_1(v), \lambda a^{-\pi_2(v)}) \neq \emptyset$. For $v \in \mathcal{S}$, we define

$$\Gamma_{k,L}(v) = \inf \left\{ \gamma = (v_1, v_2, \ldots, v_n) \;\middle|\; \begin{array}{l} \gamma \text{ is a path in } G_{\pi_2(v)+k} \text{ with } \pi_1(v_1) \in B_v, \\ \pi_1(v_n) \notin B(\pi_1(v), L a^{-\pi_2(v)}) \end{array} \right\}. \tag{4.1}$$

Define

$$\begin{aligned} M_{p,k}(L) &= \sup_{v \in \mathcal{S}} \mathrm{Mod}_p(\Gamma_{k,L}(v), G_{\pi_2(v)+k}), \\ M_p(L) &= \liminf_{k \to \infty} M_{p,k}(L). \end{aligned} \tag{4.2}$$

The *critical exponent of the combinatorial modulus* of (X, d) is defined as

$$\mathrm{CE}(X, d) = \inf\{p \in (0, \infty) : M_p(L) = 0\}. \tag{4.3}$$

If $\rho \in \mathrm{Adm}(\Gamma)$, then $1 \wedge \rho \in \mathrm{Adm}(\Gamma)$. This shows that $\mathrm{Mod}_p(\Gamma, G)$ is non-increasing in G for any family of paths Γ and any graph G. This shows that the set of p such that $M_p(L) = 0$ is an interval.

Strictly speaking $\mathrm{CE}(X, d)$ should be denoted as $\mathrm{CE}(X, d, a, \lambda, L, \{X_k : k \geq 0\})$ since it might depend on all these choices of a, λ, L and $\{X_k : k \geq 0\}$. It is known that this

[2]One could alternatively define the function ρ on edges instead of vertices but for bounded degree graphs this would lead to an equivalent quantity. This follows from an argument of He and Schramm proof in [HS, Proof of Theorem 8.1]. Our results could be stated in terms of this alternate definition as well.

exponent does not depend on the choice of $L > 1$ [Car, Lemma 3.3]. We will show that it also does not depend on the choices of the $a, \lambda \in (1,\infty)$ and $\{X_k : k \geq 0\}$. To this end, we recall the following lemma. Given a set Y, we use the notation 2^Y and $\#Y$ to denote the power set of Y and the cardinality of Y respectively.

Lemma 4.2. *[Kig22, Lemma C.4] Let $G = (V,E), \widetilde{G} = (\widetilde{V}, \widetilde{E})$ be two graphs and let $H : V \to 2^{\widetilde{V}}$ be a function so that $\#H(v) < \infty$ for all $v \in V$. Let $\Gamma, \widetilde{\Gamma}$ be two families of paths in $G, \widetilde{G}$ respectively such that for each $\gamma \in \Gamma$, there exists $\widetilde{\gamma} \in \widetilde{\Gamma}$ so that $\widetilde{\gamma}$ is contained in $\cup_{v\in\gamma} H(v)$. Then*

$$\mathrm{Mod}_p(\Gamma, G) \leq \left(\sup_{v\in V} \#H(v)\right)^p \sup_{\widetilde{v}\in\widetilde{V}} \#\{v \in V \mid \widetilde{v} \in H(v)\} \mathrm{Mod}_p(\widetilde{\Gamma}, \widetilde{G}). \tag{4.4}$$

The following proposition shows that the critical exponent for the combinatorial modulus is well defined.

Proposition 4.3 (critical exponent is well defined)**.** *Let $a, \widetilde{a}, \lambda, \widetilde{\lambda}, L, \widetilde{L} \in (1,\infty)$. Let X_k (resp. $\widetilde{X}_k$) denote a sequence of maximal a^{-k}-separated (resp. $\widetilde{a}^{-k}$-separated) subsets of X. Let $Q, \widetilde{Q}$ denote the corresponding critical exponents be as defined in* (4.3) *for these two sets of parameters. Then*

$$Q = \widetilde{Q}.$$

Proof. Let $M_{p,k}(L)$ and $\widetilde{M}_{p,k}(\widetilde{L})$ be as defined in (4.2). Let $G_k, \widetilde{G}_k, k \geq 0$ be the corresponding graphs with vertex sets $\mathcal{S}_k, \widetilde{\mathcal{S}}_k$ respectively. By symmetry, it suffices to show that $Q \leq \widetilde{Q}$. Or equivalently, it suffices to show that $Q \leq p$ for any $p > \widetilde{Q}$. To show this, we need an upper bound on $\mathrm{Mod}_p(\Gamma_{k,L}(v), G_{\pi_2(v)+k})$ for $v \in G_n, n \in \mathbb{N}$. Let $m \in \mathbb{Z}$ be the unique integer such that

$$2\widetilde{L}\widetilde{a}^{-m} \leq \frac{(L-1)}{2} a^{-1} < 2\widetilde{L}\widetilde{a}^{-m+1}. \tag{4.5}$$

For any $n \in \mathbb{N}$, let $\widetilde{n} \in \mathbb{N}$ be the unique positive integer such that

$$2\widetilde{L}\widetilde{a}^{-\widetilde{n}+(1-m)_+} \leq \frac{(L-1)}{2} a^{-n} < 2\widetilde{L}\widetilde{a}^{-\widetilde{n}+1+(1-m)_+}. \tag{4.6}$$

For any $\widetilde{k} \in \mathbb{N}$, let $k \in \mathbb{Z}$ be the unique integer such that

$$a^{-k} \leq \frac{(L-1)(\widetilde{\lambda}-1)}{4\widetilde{L}\lambda\widetilde{a}^{1+(1-m)_+}} \widetilde{a}^{-\widetilde{k}} < a^{-k+1}. \tag{4.7}$$

It is evident that there exists $k_0 \in \mathbb{N}$ such that $\widetilde{k} \geq k_0$ implies that $k \geq 1$ (that is $k \in \mathbb{N}$). For the remainder of the proof we assume $\widetilde{k} \geq k_0$. By (4.6) and (4.7), we have

$$\lambda a^{-n-k} < (\widetilde{\lambda}-1)\widetilde{a}^{-\widetilde{n}-\widetilde{k}}, \quad \frac{\widetilde{a}^{-\widetilde{n}-\widetilde{k}}}{a^{-n-k}} \leq \frac{\widetilde{a}a\lambda}{\widetilde{\lambda}-1}. \tag{4.8}$$

For all $l, \widetilde{l} \in \mathbb{N}$, we define a family of maps $H_{l,\widetilde{l}} : \mathcal{S}_l \to \widetilde{\mathcal{S}}_{\widetilde{l}}$ such that $H_{l,\widetilde{l}}(v) = \{w\}$ where $w \in \widetilde{\mathcal{S}}_{\widetilde{l}}$ is such that $d(\pi_1(v), \pi_1(w)) < \widetilde{a}^{-\widetilde{l}}$ or equivalently, $\pi_1(v) \in B(\pi_1(w), \widetilde{a}^{-\widetilde{l}})$. Since $\bigcup_{u \in \widetilde{\mathcal{S}}_{\widetilde{l}}} B(\pi_1(u), \widetilde{a}^{-l}) = X$ such a $w \in \widetilde{\mathcal{S}}_{\widetilde{l}}$ exists. By [Hei, Exercise 10.17], there exists $\beta > \dim_A(X, d)$ and $C_1 > 1$ such that

$$\sup_{w \in \widetilde{\mathcal{S}}_{\widetilde{l}}} \#\{v \in \mathcal{S}_l : w \in H(v)\} \le C_1 \left(1 \vee \frac{\widetilde{a}^{-\widetilde{l}}}{a^{-l}}\right)^{\beta}.$$

In particular, for any $k, \widetilde{k}, n, \widetilde{n} \in \mathbb{N}$ that satisfy (4.7) and (4.8), we have

$$\sup_{w \in \widetilde{\mathcal{S}}_{\widetilde{n}+\widetilde{k}}} \#\{v \in \mathcal{S}_{n+k} : w \in H(v)\} \le C_1 \left(1 \vee \frac{\widetilde{a} a \lambda}{\widetilde{\lambda} - 1}\right)^{\beta}. \tag{4.9}$$

Let $\gamma = (v_1, \ldots, v_N) \in \Gamma_{n,k}(v), v \in \mathcal{S}_n$ denote an arbitrary path. Note that, $v_i \in \mathcal{S}_{n+k}$ for all $i = 1, \ldots, N$. Consider the sequence $(w_1, \ldots, w_N)$ such that $\{w_i\} = H_{n+k, \widetilde{n}+\widetilde{k}}(v_i)$ for all $i = 1, \ldots, N$. By the first estimate in (4.8), we have $B(\pi_1(v_i), \lambda a^{-n-k}) \subset B(\pi_1(w_i), \widetilde{\lambda} \widetilde{a}^{-\widetilde{n}-\widetilde{k}})$ for all i. This in turn implies for any $i = 1, \ldots, N-1$, either $w_i = w_{i+1}$ or w_i and w_{i+1} are neighboring vertices in $\widetilde{G}_{\widetilde{n}+\widetilde{k}}$. This implies that for any $\gamma \in \Gamma_{k,L}(v)$ there exists a path $\widetilde{\gamma}$ in $\widetilde{G}_{\widetilde{n}+\widetilde{k}}$ from w_1 to w_N. Therefore by the triangle inequality,

$$d(\pi_1(w_1), \pi_1(w_N)) \ge (L-1)a^{-n} - 2\widetilde{a}^{-\widetilde{n}-\widetilde{k}} \overset{(4.6)}{\ge} (4\widetilde{L} - 2\widetilde{a}^{-\widetilde{k}})\widetilde{a}^{-\widetilde{n}} \ge 3\widetilde{L}\widetilde{a}^{-\widetilde{n}} \tag{4.10}$$

for all $\widetilde{k} \in \mathbb{N}$ large enough such that $2\widetilde{a}^{-\widetilde{k}} \le L$. Since $\pi_1(v_1) \in B(\pi_1(v), a^{-n}) \cap B(\pi_1(w_1), \widetilde{a}^{-\widetilde{n}-\widetilde{k}})$, we have $d(\pi_1(v), \pi_1(w_1)) < a^{-n} + \widetilde{a}^{-\widetilde{n}-\widetilde{k}}$. There exists $\widetilde{v} \in \widetilde{\mathcal{S}}_{\widetilde{n}}$ such that $\pi_1(w_1) \in B(\pi_1(\widetilde{v}), \widetilde{a}^{-\widetilde{n}})$. Therefore by (4.10), the path $\widetilde{\gamma} \in \widetilde{\Gamma}_{\widetilde{k},\widetilde{L}}(\widetilde{v})$ for any $\widetilde{k}$ large enough such that $2\widetilde{a}^{-\widetilde{k}} \le L$, where

$$\begin{aligned} d(\pi_1(\widetilde{v}), \pi_1(v)) &\le d(\pi_1(v), \pi_1(w_1)) + d(\pi_1(\widetilde{v}), \pi_1(w_1)) \\ &< a^{-n} + \widetilde{a}^{-\widetilde{n}-\widetilde{k}} + \widetilde{a}^{-\widetilde{n}} \le a^{-n} + 2\widetilde{a}^{-\widetilde{n}} \\ &\le \widetilde{a}^{-\widetilde{n}} \left(2 + \frac{4\widetilde{L}\widetilde{a}^{1+(1-m)_+}}{L-1}\right) \quad \text{(by (4.6))}. \end{aligned}$$

Setting $\kappa = \left(2 + \frac{4\widetilde{L}\widetilde{a}^{1+(1-m)_+}}{L-1}\right)$, we conclude that for all large enough $\widetilde{k} \in \mathbb{N}, n \in \mathbb{N}, v \in \mathcal{S}_n, \gamma \in \Gamma_{k,L}(v)$, there exists $\widetilde{\gamma} \in \widetilde{\Gamma}_{\widetilde{k},\widetilde{L}}$ such that $\widetilde{v} \in \widetilde{\mathcal{S}}_{\widetilde{n}}$ with $d(\pi_1(\widetilde{v}), \pi_1(v)) < \kappa \widetilde{a}^{-\widetilde{n}}$ and $\widetilde{\gamma}$ is contained in $\bigcup_{u \in \gamma} H_{n+k, \widetilde{n}+\widetilde{k}}(u)$, where $\widetilde{n}, k$ are as given by (4.6) and (4.7) respectively. By [Hei, Exercise 10.17], there exists $C_2 > 1$ such that

$$\sup_{v \in \mathcal{S}_n} \#\{\widetilde{v} \in \widetilde{\mathcal{S}}_{\widetilde{n}} : d(\pi_1(\widetilde{v}), \pi_1(v)) < \kappa \widetilde{a}^{-\widetilde{n}}\} \le C_2 \kappa^{\beta}. \tag{4.11}$$

Now combining the above with Lemma 4.2, (4.9) and (4.11), we obtain

$$\begin{aligned}
\operatorname{Mod}(\Gamma_{k,L}(v)) &\le C_1 \left(1 \vee \frac{\widetilde{a}a\lambda}{\widetilde{\lambda}-1}\right)^{\beta} \operatorname{Mod}\left(\bigcup_{\substack{\widetilde{v}\in\widetilde{\mathcal{S}}_{\widetilde{n}},\\ d(\pi_1(\widetilde{v}),\pi_1(v))<\kappa\widetilde{a}^{-\widetilde{n}}}} \widetilde{\Gamma}_{\widetilde{k},\widetilde{L}}(\widetilde{v}) \right) \\
&\le C_1 \left(1 \vee \frac{\widetilde{a}a\lambda}{\widetilde{\lambda}-1}\right)^{\beta} \sum_{\substack{\widetilde{v}\in\widetilde{\mathcal{S}}_{\widetilde{n}},\\ d(\pi_1(\widetilde{v}),\pi_1(v))<\kappa\widetilde{a}^{-\widetilde{n}}}} \operatorname{Mod}\left(\widetilde{\Gamma}_{\widetilde{k},\widetilde{L}}(\widetilde{v})\right) \\
&\le C_1 \left(1 \vee \frac{\widetilde{a}a\lambda}{\widetilde{\lambda}-1}\right)^{\beta} C_2\kappa^{\beta}\widetilde{M}_{p,\widetilde{k}}(\widetilde{L})
\end{aligned}$$

for all $n \in \mathbb{N}, v \in \mathcal{S}_n$ and for all $\widetilde{k} \in \mathbb{N}$ large enough. This implies that

$$M_{p,k}(L) \le C_1 \left(1 \vee \frac{\widetilde{a}a\lambda}{\widetilde{\lambda}-1}\right)^{\beta} C_2\kappa^{\beta}\widetilde{M}_{p,\widetilde{k}}(\widetilde{L})$$

for all $\widetilde{k} \in \mathbb{N}$ large enough and for all $k \in \mathbb{N}$ defined by (4.7). This immediately implies the desired inequality $Q \le p$ for any $p > \widetilde{Q}$. $\square$

The following 'reverse volume doubling estimate' is known if the metric space is uniformly perfect [Hei, Exercise 13.1]. Since our metric space is not necessarily uniformly perfect, the following lemma provides a substitute for uniform perfectness at sufficiently many scales.

Lemma 4.4. *Let μ be a doubling measure on a metric space (X, d) such that $\mu(B(x, 2r)) \le C_D\mu(B(x, r))$ for all $x \in X, r > 0$. Let $x_0, \ldots, x_N$ be a set of points such that $d(x_i, x_{i+1}) < r/4$ for all $i = 0, \ldots, N-1$ where $d(x_0, x_N) > R > r$. Then there exists $c, \alpha > 0$ depending only on C_D such that $\mu(B(x_0, R)) \ge c(R/r)^{\alpha}\mu(B(x_0, r))$.*

Proof. If $s \in [r, R/2]$, then by triangle inequality there exists x_j such that

$$\frac{5}{4}s \le d(x_0, x_j) \le \frac{7}{4}s.$$

By the doubling property, for such x_j, we have

$$\mu(B(x_j, s/4)) \ge C_D^{-3}\mu(B(x_j, 4s)) \ge C_D^{-3}\mu(B(x_0, s))$$

Therefore

$$\mu(B(x_0, 2s)) \ge \mu(B(x_0, s)) + \mu(B(x_j, s/4)) \ge (1 + C_D^{-3})\mu(B(x_0, s))$$

for all such $s \in [r, R/2]$. Let k be the largest integer such that $2^k r \le R$. By iterating the above estimate

$$\mu(B(x_0, R)) \ge \mu(B(x_0, 2^{k-1}r)) \ge (1 + C_D^{-3})^{k-1}\mu(B(x_0, r)) \ge c\left(\frac{R}{r}\right)\mu(B(x_0, r))$$

where $\alpha = \log(1 + C_D^{-3})/\log 2$ and $c = (1 + C_D^{-3})^{-1}$. $\square$

4.1 Proof of Theorem 1.2

In this section, we complete the proof of Theorem 1.2. That is, we show that for any compact, doubling, metric space (X, d), we have

$$\dim_{\mathrm{CA}}(X,d) = \mathrm{CE}(X,d) = \inf\{\dim_{\mathrm{A}}(X,\theta) : \theta \in \mathcal{J}_p(X,d)\}.$$

The following lemma is useful to obtain upper bounds on the critical exponent $\mathrm{CE}(X, d)$.

Lemma 4.5. *Let (X, d) be a compact metric space and $a \geq \lambda \geq 6$. Let $\theta \in \mathcal{J}(X, d)$ and μ be a q-homogeneous measure on (X, θ). Let $\mathcal{S}$ denote a hyperbolic filling with parameters a, λ. For all $v \in \mathcal{S}, k \in \mathbb{N}$, we define $\rho_v : \mathcal{S}_{\pi_2(v)+k} \to [0, \infty)$ as*

$$\rho_v(w) = \begin{cases} \left(\frac{\mu(B_w)}{\mu(B_v)}\right)^{1/q} & \text{if } B_w \cap B_d(\pi_1(v), (L+1)a^{-\pi_2(v)}) \neq \emptyset,\ w \in \gamma \text{ for some } \gamma \in \Gamma_{k,L}(v), \\ 0 & \text{otherwise.} \end{cases} \tag{4.12}$$

Then there exists $c > 0, k_0 \in \mathbb{N}$ depending only on d, θ, μ, a, L so that

$$\sum_{w\in\gamma} \rho_v(w) \geq c \quad \text{for all } \gamma \in \Gamma_{k,L}(v), v \in \mathcal{S} \tag{4.13}$$

for all $k \geq k_0$.

Proof. Let $\gamma \in \Gamma_{k,L}(v)$. To show (4.13), by choosing a sub-path if necessary, we may assume that $\gamma = (v_1, \ldots, v_N)$ and $\pi_1(v_1) \in B_v$, $\pi_1(v_j) \notin B(\pi_1(v), a^{-\pi_2(v)}) = B_v$ for all $j = 2, \ldots, N-1$, $\pi_1(v_i) \in B(\pi_1(v), La^{-\pi_2(v)})$ for all $i = 1, \ldots, N-1$ and $v_N \notin \pi_1(v_i) \in B_d(\pi_1(v), La^{-\pi_2(v)})$. Since

$$d(\pi_1(v_i), \pi_1(v_{i+1})) < 2\lambda a^{-k-\pi_2(v)} \leq a^{-k-\pi_2(v)+1} \leq a^{-\pi_2(v)} \quad \text{for all } i = 1, \ldots, N-1, \tag{4.14}$$

we have $v_N \in B(v, (L+1)a^{-\pi_2(v)})$. In particular,

$$z_i \in B_d(\pi_1(v_i), 2\lambda a^{-\pi_2(v_i)}) \setminus B_d(\pi_1(v_i), a^{-\pi_2(v_i)}) \neq \emptyset \quad \text{for all } i = 1, \ldots, N, \tag{4.15}$$

where $z_i = \pi_1(v_{i+1})$ for $i = 1, \ldots, N-1$ and $\pi_1(v_{i-1})$ for $i = N$. Let $\eta : [0, \infty) \to [0, \infty)$ be a distortion function such that the identity map $\mathrm{Id} : (X, d) \to (X, \theta)$ is an η-quasisymmetry. This along with the choice of z_i above this implies that

$$B_\theta(\pi_1(v_i), c_1\theta(\pi_1(v_i), z_i)) \subset B_d(\pi_1(v_i), a^{-\pi_2(v_i)}) = B_{v_i} \subset B_\theta(\pi_1(v_i), \eta(1)\theta(\pi_1(v_i), z_i)) \tag{4.16}$$

for all $i = 1, \ldots, N$, where $c_1 = [\eta(2\lambda)]^{-1}$. Since $d(\pi_1(v), \pi_1(v_i)) < (L+1)a^{-\pi_2(v)}$ for all $i = 1, \ldots, N$, we have $B_v = B_d(\pi_1(v), a^{-\pi_2(v)}) \subset B_d(\pi_1(v_i), (L+2)a^{-\pi_2(v)})$. Choosing $w_i \in \{\pi_1(v), \pi_1(v_N)\}$ such that $2(L+1)a^{-\pi_2(v)} > d(\pi_1(v_i), w_i) \geq d(\pi_1(v), \pi_1(v_N))/2 \geq La^{-\pi_2(v)}/2$, we have

$$B_v \subset B_d(\pi_1(v_i), (L+2)a^{-\pi_2(v)}) \subset B_\theta(\pi_1(v_i), C_2\theta(\pi_1(v_i), w_i)) \tag{4.17}$$

for all $i = 1, \dots, N$, where $C_2 = \eta(2(L+2)L^{-1})$. Furthermore

$$B_\theta(\pi_1(v_i), \theta(\pi_1(v_i), w_i)) \subset B_d(\pi_1(v_i), \eta(1)d(\pi_1(v_i), w_i)) \subset B_d(\pi_1(v), C_3 a^{-\pi_2(v)}), \quad (4.18)$$

where $C_3 = (L+1)(2\eta(1)+1)$.

Since μ is q-homogeneous on (X, θ) and $\theta \in \mathcal{J}(X, d)$, μ is a doubling measure on (X, d). Therefore

$$\mu(B_v) \gtrsim \mu\left(B_d(\pi_1(v), C_3 a^{-\pi_2(v)})\right) \geq \mu(B_\theta(\pi_1(v_i), \theta(\pi_1(v_i), w_i))). \quad (4.19)$$

Since $d(\pi_1(v_i), \pi_1(v_{i+1}))/d(\pi_1(v_i), w_i) \leq 4L^{-1}\lambda a^{-k}$

$$\theta(\pi_1(v_i), \pi_1(v_{i+1})) \leq \theta(\pi_1(v_i), w_i)\eta\left(4L^{-1}\lambda a^{-k}\right) \quad \text{for all } i = 1, \dots, N-1. \quad (4.20)$$

Pick $k_0 \in \mathbb{N}$ large enough so that

$$c_1\eta\left(4L^{-1}\lambda a^{-k}\right) \leq 1. \quad (4.21)$$

Let $k \geq k_0$. Since μ is a q-homogeneous measure on (X, θ) and $\mathrm{Id} : (X, d) \to (X, \theta)$ is an η-quasisymmetry, we have

$$\begin{aligned}
\sum_{i=1}^{N} \rho_v(v_i) &= \sum_{i=1}^{N} \left(\frac{\mu(B_{v_i})}{\mu(B_v)}\right)^{1/q} \\
&\gtrsim \sum_{i=1}^{N-1} \left(\frac{\mu(B_\theta(\pi_1(v_i), c_1\theta(\pi_1(v_i), \pi_1(v_{i+1}))))}{\mu\left(B_\theta(\pi_1(v_i), \theta(\pi_1(v_i), w_i))\right)}\right)^{1/q} \quad \text{(by (4.16) and (4.17))} \\
&\gtrsim \sum_{i=1}^{N-1} \frac{\theta(\pi_1(v_i), \pi_1(v_{i+1}))}{\theta(\pi_1(v_i), w_i)} \quad \text{(by (4.21) and } q\text{-homogeneity of } \mu \text{ in } (X, \theta)) \\
&\gtrsim \sum_{i=1}^{N-1} \frac{\theta(\pi_1(v_i), \pi_1(v_{i+1}))}{\theta(\pi_1(v), \pi(v_N))} \quad (\text{since } d(\pi_1(v_i), w_i) \lesssim d(\pi_1(v), \pi_1(v_N))) \\
&\gtrsim \frac{\theta(\pi_1(v_1), \pi(v_N))}{\theta(\pi_1(v), \pi(v_N))} \gtrsim 1 \\
&\quad \text{(by triangle inequality and } d(\pi_1(v_1), \pi(v_N)) \gtrsim d(\pi_1(v), \pi(v_N))).
\end{aligned}$$

This completes the proof of (4.13), where $c > 0$ depends only on η, q-homogeneity constants of μ and λ, a, L. $\square$

Proof of Theorem 1.2. By Proposition 4.3, it suffices to consider the critical exponent $a \geq \lambda \geq 6$, where the maximal a^{-n} separated subsets are increasing (similar to the definition of hyperbolic filling).

The inequality $\dim_{\mathrm{CA}}(X, d) \leq \mathrm{CE}(X, d)$ follows from the same argument as the proof of $\dim_{\mathrm{ARC}}(X, d) \leq \mathrm{CE}(X, d)$ in [Car, Theorem 1.3] where the use of [Car, Theorem 1.2] is replaced with Theorem 3.4. This yields the inequality

$$\dim_{\mathrm{CA}}(X, d) \leq \inf\{\dim_{\mathrm{A}}(X, \theta) : \theta \in \mathcal{J}_p(X, d)\} \leq \mathrm{CE}(X, d). \quad (4.22)$$

So it suffices to show $\mathrm{CE}(X,d) \le \dim_{\mathrm{CA}}(X,d)$. Let $p > \dim_{\mathrm{CA}}(X,d)$. We consider a hyperbolic filling $\mathcal{S}$ of (X,d) with parameters $a \ge \lambda \ge 6$. Then by Theorem 3.1, for any $q \in (\dim_{\mathrm{CA}}(X,d), p)$, there exists $\theta \in \mathcal{J}(X,d)$ and a q-homogeneous measure μ on (X,θ).

Next, we show the following estimate: there exists $C > 0, k_0 \in \mathbb{N}$ such that

$$\operatorname{Mod}_p\left(\Gamma_{k,L}(v)\right) \le C a^{-k\alpha(p-q)} \quad \text{for any } v \in \mathcal{S}, k \in \mathbb{N} \text{ with } k \ge k_0. \tag{4.23}$$

Let $\rho_v : \mathcal{S}_{\pi_2(v)+k} \to [0,\infty)$ be as given in Lemma 4.5. By Lemma 4.5, it suffices to estimate $\sum_w \rho_v(w)^p$.

To this end, we first obtain an upper bound on $\rho_v(w)$. For any $v \in \mathcal{S}$, let $w \in \mathcal{S}_{\pi_2(v)+k}$ such that $w \in \gamma$ for some $(v_1,\ldots,v_N) = \gamma \in \Gamma_{k,L}(v)$. Note that $d(\pi_1(w),\pi_1(v_1)) \vee d(\pi_1(w),\pi_1(v_N)) \ge \frac{1}{2} d(\pi_1(v_1),\pi_1(v_N)) \ge (L-1)a^{-\pi_2(v)}/2$. Therefore there exists a sequence $w = x_0,\ldots,x_M$ so that $d(x_0,x_M) \ge (L-1)a^{-\pi_2(v)}$. Choose $k_1 \in \mathbb{N}$ such that $(L-1)a^{-k_1} < \frac{1}{4}$. By the volume doubling property of μ on (X,d), we have $\mu(B_v) \gtrsim \mu(B(\pi_1(v), 2La^{-\pi_2(v)})) \gtrsim \mu(B(\pi_1(w),(L-1)a^{-\pi_2(v)/2}))$. For all $k \ge k_1$, by Lemma 4.4, we have

$$\frac{\mu(B_w)}{\mu(B_v)} \lesssim \frac{\mu(B_w)}{\mu(B(\pi_1(w),(L-1)a^{-\pi_2(v)}/2))} \le C a^{-\alpha k} \quad \text{for all } v \in \mathcal{S}, w \in \gamma, \gamma \in \Gamma_{k,L}(v), \tag{4.24}$$

where C,α only depends on λ, a, L, and the doubling constant of μ in (X,d). Since μ is a doubling measure, we have

$$\begin{aligned}
\sum_{w \in \mathcal{S}_{\pi_2(v)+k}} \rho_v(w)^q &\le \sum_{\substack{w \in \mathcal{S}_{\pi_2(v)+k},\\ \pi_1(w) \in B(\pi_1(v),(L+1)a^{-\pi_2(v)})}} \frac{\mu(B_w)}{\mu\left(B_d(\pi_1(v),(L+2)a^{-\pi_2(v)})\right)} \\
&\lesssim \sum_{\substack{w \in \mathcal{S}_{\pi_2(v)+k},\\ \pi_1(w) \in B(\pi_1(v),(L+1)a^{-\pi_2(v)})}} \frac{\mu(B_d(\pi_1(w),a^{-\pi_2(w)}/2))}{\mu\left(B_d(\pi_1(v),(L+2)a^{-\pi_2(v)})\right)} \\
&\quad \text{(since } \mu \text{ is doubling)} \\
&\lesssim 1 \quad \text{(since } B_d(\pi_1(w),a^{-\pi_2(w)}/2)) \text{ pairwise disjoint)}.
\end{aligned} \tag{4.25}$$

By Lemma 4.5, there exists $c > 0, k_0 \in \mathbb{N}$ such that $c^{-1}\rho_v \in \operatorname{Adm}(\Gamma_{k,L}(v))$ for al $k \ge k_0$. Hence by (4.24) and (4.25), we have

$$\operatorname{Mod}_p(\Gamma_{k,L}(v)) \lesssim \sum_w \rho_v(w)^p \le \left(\sup_w \rho_v(w)\right)^{p-q} \sum_w \rho_v(w)^q \lesssim a^{-k\alpha(p-q)} \quad \text{for all } k \ge k_0.$$

This concludes the proof of (4.23) and hence we obtain $M_{p,k}(L) \lesssim a^{-k\alpha(p-q)}$ for all $k \ge k_0$. This shows $M_p(L) = 0$ and hence $\mathrm{CE}(X,d) \le p$ for all $p > \dim_{\mathrm{CA}}(X,d)$. This along with (4.22) concludes the proof. ☐

One might wonder if the assumption $\dim_{\mathrm{A}}(X,d) < \infty$ (or equivalently, (X,d) is a doubling metric space) in Theorem 1.2 is necessary. To this end, we present the following example.

Example 4.6. Let X denote the set of all sequences $(x_i)_{i\in\mathbb{N}}$ such that $x_i \in \{1, 2, \ldots, i\}$; that is, $X = \prod_{i=1}^{\infty}\{1, \ldots, i\}$. We define a metric on X by setting

$$d((x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}}) = \begin{cases} 0 & \text{if } x_i = y_i \text{ for all } i \in \mathbb{N}, \\ 2^{-j} & \text{if } j = \min\{k : x_k \neq y_k\} < \infty. \end{cases}$$

It is easy to see that (X, d) is a compact, ultrametric space. Since every open ball of radius 2^{-k} has $k+1$ distinct points with mutual distance of at least 2^{-k-1} for each $k \in \mathbb{N}$, we have

$$\dim_{\mathrm{A}}(X, d) = \infty. \tag{4.26}$$

On the other hand, consider $a, \lambda, L > 1, X_k, \mathcal{S}_k$ as given in Definition 4.1. For any $v \in \mathcal{S}_n$ and any $\gamma = (v_1, \ldots, v_N) \in \Gamma_{k,L}(v)$, since (X, d) is an ultrametric space, we have

$$(L-1)a^{-n} \leq d(\pi_1(v_1), \pi_1(v_N)) \leq \max_{1\leq i\leq N-1} d(\pi_1(v_i), \pi_1(v_{i+1})) < 2\lambda a^{-n-k}.$$

Therefore for all k large enough so that $a^{-k} \leq \frac{L-1}{2\lambda}$, we have $\Gamma_{k,L}(v) = \emptyset$ and hence $M_{p,k}(L) = 0$ for all $p > 0$ and k large. This implies $\mathrm{CE}(X, d) = 0$. Therefore by (4.26), we have

$$\dim_{\mathrm{CA}}(X, d) = \infty \neq 0 = \mathrm{CE}(X, d).$$

Therefore the assumption that (X, d) is a doubling metric space is necessary in Theorem 1.2.

We conclude with some questions about the closely related notion of conformal (Hausdorff) dimension. Recall that the conformal (Hausdorff) dimension $\dim_{\mathrm{CH}}(X, d)$ is defined as

$$\dim_{\mathrm{CH}}(X, d) = \inf\{\dim_{\mathrm{H}}(X, \theta) : \theta \in \mathcal{J}(X, d)\},$$

where $\dim_{\mathrm{H}}(X, \theta)$ denotes the Hausdorff dimension of (X, θ). Does the equality

$$\dim_{\mathrm{CH}}(X, d) = \inf\{\dim_{\mathrm{H}}(X, \theta) : \theta \in \mathcal{J}_p(X, d)\}$$

always hold? Theorem 1.2 shows a similar result for the Ahlfors regular conformal dimension. It is also interesting to know for which metric spaces does $\dim_{\mathrm{CH}}(X, d) = \dim_{\mathrm{CA}}(X, d)$ hold? It is easy to see that $\dim_{\mathrm{CH}}(X, d) \leq \dim_{\mathrm{CA}}(X, d)$. One might expect that for 'self-similar sets' like the standard Sierpinski carpet $\dim_{\mathrm{CH}}(X, d) = \dim_{\mathrm{CA}}(X, d)$ holds. This seems to be a difficult problem since 'self-similarity' is not a quasisymmetry invariant. It is not known whether the equality $\dim_{\mathrm{CH}}(X, d) = \dim_{\mathrm{CA}}(X, d)$ holds even for the standard Sierpinski carpet.

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