Homework 9 - Math 321, Spring 2015

1. *Let f be a differentiable 2π-periodic function with continuous first derivative. Is f the uniform limit of its partial Fourier sums?*

   Solution. Yes. Using integration by parts and periodicity of f,

$$\widehat{f'(k)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(t)e^{-ikt}dt = \frac{1}{2\pi}f(t)e^{-ikt}\Big|_{-\pi}^{\pi} + \frac{ik}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ikt}dt$$
$$= ik\hat{f}(k)$$

   By the Cauchy-Schwarz inequality,

$$\Big(\sum_{k\in\mathbb{Z}} |\widehat{f(k)}|\Big)^2 \leq \Big(\sum_{k\in\mathbb{Z}} \frac{1}{k^2}\Big)\Big(\sum_{k\in\mathbb{Z}} k^2|\widehat{f(k)}|^2\Big)$$

   The series $\sum_{k\in\mathbb{Z}} \frac{1}{k^2}$ is convergent. By Parseval's identity and continuity of f',

$$\sum_{k\in\mathbb{Z}} k^2|\widehat{f(k)}|^2 = \sum_{k\in\mathbb{Z}} |\widehat{f'(k)}|^2 = \|f'\|_2^2 < \infty.$$

   Hence $\sum_k |\widehat{f}(k)| < \infty$. Since f is continuous, it follows from a result proved in class that its partial fourier sum converges uniformly to f.

   $\square$

2. *Determine whether the following statement is true or false: If $f : \mathbb{R} \to \mathbb{R}$ is 2π-periodic and Riemann-integrable on $[-\pi, \pi]$, then $||f_\epsilon - f||_2 \to 0$ as $\epsilon \to 0$. Here f_ϵ denotes the translated function $f_\epsilon(x) = f(x + \epsilon)$.*

   Solution. We employ a density argument. If g is assumed to be continuous and 2π-periodic on $[-\pi, \pi]$, it follows from uniform continuity (write out a proof of this!) that

$$\lim_{\epsilon\to 0} \sup_{x\in[-\pi,\pi]} \big|g_\epsilon(x) - g(x)\big| = 0, \quad \text{hence} \quad ||g_\epsilon - g||_2 \leq ||g_\epsilon - g||_\infty \to 0, \tag{1}$$

   so the result is verified for continuous functions.

   Now let f be a general 2π-periodic Riemann integrable function on $[-\pi, \pi]$. Then by problem 4 of HW set 8, for any given $\kappa > 0$, there exists a continuous and 2π-periodic function g on $[-\pi, \pi]$ such that $||f - g||_2 < \frac{\kappa}{2}$. In particular, this means that $||f_\epsilon - g_\epsilon||_2 < \frac{\kappa}{2}$. Therefore, using the first part of the argument, namely (1), we find that

$$||f_\epsilon - f||_2 \leq ||f_\epsilon - g_\epsilon||_2 + ||f - g||_2 + ||g_\epsilon - g||_2 \leq \kappa \quad \text{as } \epsilon \to 0.$$

   Since κ was arbitrary, this completes the proof of the theorem. $\square$

3. (a) *Obtain the Fourier series of the 2π-periodic function that coincides with $f(t) = (\pi - t)^2$ on $[0, 2\pi]$.*

   (b) *Does f match its Fourier series? Give reasons for your answer.*

   (c) *Use your results from above to derive the identity: $\sum_{n=1}^{\infty} n^{-2} = \frac{\pi^2}{6}$.*

Solution. Using Fourier expansion

$$(\pi - t)^2 = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kt) + b_k \sin(kt)$$

where

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi - t)^2 dt = \frac{8\pi^2}{3} \\ a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi - t)^2 \cos(kt) dt = \frac{4(\pi^2 k^2 - 1)\sin(k\pi) + \pi k}{\pi k^3} \\ b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi - t)^2 \sin(kt) dt = \frac{4\pi(\pi k \cos(k\pi)) - \sin(k\pi)}{\pi k^2} \end{aligned}$$

The series converges uniformly to the function since it is differentiable with continuous derivative using Question 1 (one may only have to check this at $t = 0$ using definition of derivative). Putting $t = \pi$ in the fourier series gives $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$

□

4. (a) *Show that the Fourier series of a function f can alternatively be written in the form*

$$\sum_{k=-\infty}^{\infty} \widehat{f}(k) e^{ikx}, \qquad \textit{where} \qquad \widehat{f}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) e^{-ikt}\, dt$$

is referred to as the kth Fourier coefficient.

(b) *Determine the relation of $\widehat{f}(k)$ with $\widehat{g}(k)$ in each of the following cases:*

(i) *g is a translate of f, namely $g(x) = f(x + \alpha)$.*

(ii) *g is a modulation of f, namely $g(x) = f(x)e^{-i\alpha x}$.*

(c) *Given two bounded, 2π-periodic functions f and g, both of which are Riemann-integrable on $[-\pi, \pi]$, define their convolution $h = f * g$ as follows,*

$$h(x) = f * g(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x - t) g(t)\, dt.$$

Note that the nth partial Fourier sum $s_n f$ and the nth Cesàro sum $\sigma_n f$ are both given in terms of convolutions of f with appropriate kernels. Find $\widehat{h}$ in terms of $\widehat{f}$ and $\widehat{g}$.

(d) *Use this and the Fourier coefficients of K_n to derive an explicit formula for the nth Cesàro sum $\sigma_n f$ of a Fourier series.*

Solution. (a) Substitute $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$ and $\cos x = \frac{e^{ix} + e^{-ix}}{2}$ into the the partial fourier sum (also in a_k, b_k)

$$\frac{a_0}{2} + \sum_{k \geq 1} a_k \cos(kx) + \sum_{k \geq 1} b_k \sin(kx)$$

To see that it is

$$\sum_{|k| \leq n} c_k e^{ikx}, c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) e^{-ikt} dt$$

(b) If $g(x) = f(x+\alpha)$ then

$$\hat{g}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+\alpha)e^{-ikx}dx = e^{ik\alpha}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+\alpha)e^{-ik(x+\alpha)}dx$$

Since f is 2π-periodic, this becomes

$$e^{i\alpha k}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-iyk}dy = e^{i\alpha k}\hat{f}(k)$$

If $g(x) = f(x)e^{-i\alpha x}$ them

$$\hat{g}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-i\alpha x}e^{-ikx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-i(\alpha+k)x}dx = \hat{f}(k+\alpha)$$

(c) Since f, g are 2π periodic,

$$\begin{aligned}
\widehat{f*g}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2\pi}\left(\int_{-\pi}^{\pi} f(y)g(x-y)dy\right)e^{-inx}dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-iny}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} g(x-y)e^{-in(x-y)}dx\right)dy \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-iny}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)e^{-inx}dx\right)dy \\
&= \hat{f}(n)\hat{g}(n)
\end{aligned}$$

(d) We know from definition that

$$K_n(t) = \sum_{|j|\le n}\left(1-\frac{|j|}{n}\right)e^{ijt}$$

Hence

$$\widehat{K_n}(j) = \begin{cases} 1-\frac{|j|}{n} & \text{if } |j|\le n \\ 0 & \text{if } |j| > n. \end{cases}$$

By the previous part of the exercise

$$\widehat{\sigma_n} = \widehat{f*K_n}(j) = \begin{cases} \left(1-\frac{|j|}{n}\right)\widehat{f}(j) & \text{if } |j|\le n \\ 0 & \text{if } |j| > n. \end{cases}$$

By uniqueness of Fourier series (in this case they are trigonometric polynomials)

$$\sigma_n(t) = \sum_{|j|\le n}\left(1-\frac{|j|}{n}\right)\widehat{f}(j)e^{ijt}$$

$\square$

5. *Let $f \in \mathcal{C}^{2\pi}$, the class of continuous 2π-periodic functions on $[-\pi,\pi]$. We have seen that the partial Fourier sums $s_n(f)$ are excellent approximations of f in L^2 norm, but so far only have indirect evidence that they may not be very good approximations in the sup norm. This problem attempts to make this intuition precise.*

(a) *Show that for every $n \geq 1$, there exists $f_n \in \mathcal{C}^{2\pi}$ such that $||f_n||_\infty = 1$ and $\sup_j |s_j f_n(0)| > n$.*

(b) *Use the functions f_n in part (a) to find a single $f \in \mathcal{C}^{2\pi}$ whose Fourier series diverges at 0.*

(c) *Now modifiy your construction in part (b) to create a continuous 2π-periodic function whose Fourier series diverges at a dense set of points.*

Solution. (a) Let $g_j = \mathrm{sgn}(D_j)$, where D_j is the Dirichlet kernel. We have seen in class that

$$s_j g_j(0) = \frac{1}{\pi}\int_{-\pi}^{\pi} |D_j(t)|\, dt \geq \frac{4}{\pi^2}\log j. \tag{2}$$

Let $\mathcal{G}_j$ denote the set of discontinuities of the function g_j, along with the endpoints $\pm\pi$. By the nature of the function D_j, the set $\mathcal{G}_j$ consists of finitely many points, say $\mathcal{G}_j = \{x_1 = -\pi < x_2 < \cdots < x_{N_j} = \pi\}$. Picking $\epsilon_j > 0$ to be a constant small enough so that

$$\epsilon_j < \frac{1}{2}\min\{|x_k - x_\ell| : x_k \neq x_\ell; x_k, x_\ell \in \mathcal{G}_j\} \quad \text{and} \quad \frac{4}{\pi}N_j\epsilon_j(j+\frac{1}{2}) < \frac{2}{\pi^2}\log j,$$

let us define a continuous function h_j as follows: h_j is linear on the interval $I_k = (x_k - \epsilon_j, x_k + \epsilon_j)$ for each $k = 1, \cdots, N_j$, and $h_j = g_j$ otherwise. Then $||h_j||_\infty = 1$ for each j. Hence $|g_j - h_j| \leq 2$ everywhere, from which we obtain the estimate

$$\begin{aligned}
|s_j h_j(0) - s_j g_j(0)| &\leq \frac{1}{\pi}\int_{-\pi}^{\pi} |D_j(y)||g_j(y) - h_j(y)|\, dy \\
&\leq \frac{1}{\pi}||D_j||_\infty \sum_k \text{length of } I_k \\
&\leq \frac{1}{\pi}(N_j)(2\epsilon_j)(2)(j+\frac{1}{2}) < \frac{2}{\pi^2}\log j, \qquad (3)
\end{aligned}$$

where we have used the fact that $||D_j||_\infty \leq j + \frac{1}{2}$ at the second step. Combining (2) and (3) we obtain that $|s_j h_j(0)| \geq |s_j g_j(0)| - |s_j(h_j - g_j)(0)| \geq \frac{2}{\pi^2}\log j$. Picking the index $j = j_n$ so that $\frac{2}{\pi^2}\log j > n$ and setting $f_n := h_{j_n}$ gives the desired result.

(b) Without loss of generality, after convolving with an appropriate Fejér kernel if necessary (verify this!), we may assume that each f_n in part (a) is a trigonometric polynomial, say of degree d_n, which increases with n. Let us pick a fast-increasing sequence of integers M_n and a fast-decaying sequence of positive numbers $\epsilon_n < 2^{-n}$ obeying the following rules: for every $n \geq 1$,

$$M_n > d_{n-1}, \qquad \epsilon_n M_n > 2^n, \qquad \epsilon_n(2M_{n-1}+1) < 2^{-n-1}.$$

Set $f = \sum_n \epsilon_n f_{M_n}$. The condition $||f_n||_\infty \leq 1$ ensures that the series converges uniformly, hence f is a continuous function. Further, for every $n \geq 1$,

$$\begin{aligned}
\left|s_{M_n} f(0)\right| &\geq \epsilon_n \left|s_{M_n} f_{M_n}(0)\right| - \sum_{k\neq n} \epsilon_k s_{M_n} f_{M_k}(0) \\
&\geq \epsilon_n M_n - \sum_{k<n} \epsilon_k |f_{M_k}(0)| - 2\sum_{k>n} \epsilon_k(2M_n+1) \\
&\geq 2^n - \sum_{k<n} 2^{-k} - \sum_{k>n} 2^{-k}
\end{aligned}$$

$$\geq 2^n - 1.$$

The second step above uses the fact that $s_{M_n} f_{M_k} = f_{M_k}$ if $k < n$, since $d_k < M_n$. We have also used the fact that $||f_n||_\infty = 1$, from which it follows that $||s_k f_n||_\infty \leq 2(2k+1)$, since the absolute value of every Fourier coefficient is at most 2. Thus we conclude that the sequence $\{s_j f(0) : j \geq 1\}$ diverges.

(c) Let $\{q_n\}$ be an enumeration of the rationals in $[-\pi, \pi]$. Using part (b), first find functions φ_n such that the sequence $\{s_j \varphi_n(q_n) : j \geq 1\}$ diverges. Now use an argument similar to (b) to construct φ meeting the specifications of part (c).

□

6. *This exercise is designed to study a curious property of a certain class of Fourier series, known as Gibbs phenomenon. Discovered by Wilbraham (1849) and studied by Gibbs (1899), this phenomenon refers to the manner in which the Fourier series of a piecewise continuously differentiable periodic function behaves at a jump discontinuity. The nth partial Fourier sums oscillate near the jump point, which is understandable, but the strange thing is that the oscillation might result in increasing the maximum of the partial sum above that of the function itself. Even more strange is the fact that the overshoot does not die out as you take larger and larger sums (i.e. the frequency increases), but approach a finite nonzero limit! Here is an example where you can see Gibbs phenomenon in action.*

Let $f(x) = sgn(x)$, which takes on the value 1, -1 or 0 according as x is positive, negative or zero.

(a) *Show that*

$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)} \qquad \text{for every } x \in [-\pi, \pi].$$

(b) *Denote by s_n the nth partial sum of the above series. Show that*

$$s_n(x) = \frac{2}{\pi} \int_0^x \frac{\sin 2nt}{\sin t}\, dt.$$

(c) *Examine the local maxima and minima of s_n, and deduce that the largest value of s_n is attained at $\frac{\pi}{2n}$.*

(d) *Interpret $s_n(\frac{\pi}{2n})$ as a Riemann sum and prove that*

$$\lim_{n\to\infty} s_n\left(\frac{\pi}{2n}\right) = \frac{2}{\pi} \int_0^\pi \frac{\sin t}{t}\, dt.$$

The value of this limit is about 1.179. Thus, although f has a jump equal to 2 at the origin, the graphs of the approximating curves s_n tend to approximate a vertical segment of length 2.358 in the vicinity of the origin!

Solution. (a) Verify that the series on the right is the Fourier series of f, so it remains to verify that the series converges pointwise to $f(x)$ for every $x \in [-\pi, \pi]$. Clearly, equality holds for $x = 0$. For any $x \neq 0$,

$$f * D_n(x) - f(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} \big[f(x-y) - f(x)\big] \frac{\sin(n+\frac{1}{2})y}{\sin\left(\frac{y}{2}\right)}\, dy$$

$$= -\frac{2}{\pi}\int_I \frac{\sin(n+\frac{1}{2})y}{\sin\left(\frac{y}{2}\right)}\,dy$$

where I is the union of two intervals not containing 0. Integrating by parts with $u = 1/\sin(y/2)$ and $dv = \sin((n+\frac{1}{2})y)$ leads to the integral being bounded from above by C/n, where C is a constant depending on n. Thus $f * D_n(x) \to f(x)$ as $n \to \infty$, which is the desired conclusion.

(b) We estimate s_n as follows,

$$\begin{aligned} s_n(x) &= \frac{4}{\pi}\int_0^x \sum_{k=1}^n \cos(2k-1)t\,dt \\ &= \frac{4}{\pi}\int_0^x \frac{\sin 2nt}{\sin t}\,dt, \end{aligned}$$

since $\sin t\cos(2k-1)t = \sin 2kt - \sin(2k-2)t$.

(c) Since $s_n'(x) = \frac{4}{\pi}\frac{\sin 2nx}{\sin x}$, the critical points of s_n in $[-\pi,\pi]$ are $x = \pm\frac{k\pi}{2n}$ for $k = 1,\cdots,(2n-1)$. Further,

$$\frac{\pi}{4}s_n''(\pm\frac{k\pi}{2n}) = 2n\frac{\cos k\pi}{\sin\left(\pm\frac{k\pi}{n}\right)} = \frac{2n(-1)^k}{\sin\left(\frac{\pm\pi k}{2n}\right)},$$

from which we conclude that the local maxima are attained at $x = \frac{\pi}{2n}, -\frac{2\pi}{2n}, \frac{3\pi}{2n}, -\frac{4\pi}{2n}\cdots$. The other critical points are local minima.

Since the function $s_n(x)$ is odd, the largest value of s_n is the maximum of $|s_n(\frac{(2k-1)\pi}{2n})|$ for $k = 1,\cdots,n$. Suppose further that $2k+1 \le n$; then,

$$\begin{aligned} s_n\left((2k+1)\frac{\pi}{2n}\right) - s_n\left((2k-1)\frac{\pi}{2n}\right) &= \frac{4}{\pi}\int_{\frac{(2k-1)\pi}{2n}}^{\frac{(2k+1)\pi}{2n}} \frac{\sin 2nt}{\sin t}\,dt \\ &= \frac{4}{\pi}\int_{(2k-1)\pi}^{(2k+1)\pi} \frac{\sin s}{\sin\left(\frac{s}{2n}\right)}\,ds \\ &= \frac{4}{\pi}\Big[\int_{(2k-1)\pi}^{2k\pi} \frac{\sin s}{\sin\left(\frac{s}{2n}\right)}\,ds + \int_{2k\pi}^{(2k+1)\pi} \frac{\sin s}{\sin\left(\frac{s}{2n}\right)}\,ds\Big] \\ &= \frac{4}{\pi}\Big[\int_{(2k-1)\pi}^{2k\pi} \frac{\sin s}{\sin\left(\frac{s}{2n}\right)}\,ds - \int_{(2k-1)\pi}^{2k\pi} \frac{\sin u}{\sin\left(\frac{\pi+u}{2n}\right)}\,du\Big] \\ &= \frac{4}{\pi}\int_{(2k-1)\pi}^{2k\pi} \sin s\left[\frac{1}{\sin\left(\frac{s}{2n}\right)} - \frac{1}{\sin\left(\frac{\pi+s}{2n}\right)}\right]ds \\ &\le 0, \end{aligned}$$

where the last step follows from the fact that the first factor of the integrand is non-positive, while the second one is non-negative on the domain of integration. This in turn is a consequence of the assumption that both $\frac{s}{n}, \frac{\pi+s}{n} \in [0,\frac{\pi}{2}]$, and the non-decreasing nature of the function $\sin x$ in the range $[0,\frac{\pi}{2}]$. On the other hand, if $2k-1 \ge n$, then setting $k = n-r$, we obtain that $2r+1 \le n$. The same set of calculations as above now yields that

$$s_n\left((2k+1)\frac{\pi}{2n}\right) - s_n\left((2k-1)\frac{\pi}{2n}\right) = \frac{4}{\pi}\int_{(2k-1)\pi}^{2k\pi} \sin s\left[\frac{1}{\sin\left(\frac{s}{2n}\right)} - \frac{1}{\sin\left(\frac{\pi+s}{2n}\right)}\right]ds$$

$$
\begin{aligned}
&\leq \int_{(2k-1)\pi}^{2k\pi} \sin s \left[\frac{1}{\sin\left(\frac{s}{2n}\right)} - \frac{1}{\sin\left(\frac{\pi+s}{2n}\right)} \right] ds \\
&\leq \int_{2r\pi}^{(2r+1)\pi} \sin t \left[\frac{1}{\sin\left(\frac{2n\pi-t}{2n}\right)} - \frac{1}{\sin\left(\frac{\pi+2n\pi-t}{2n}\right)} \right] dt \\
&\int_{2r\pi}^{(2r+1)\pi} \sin t \left[\frac{1}{\sin\left(\frac{t}{2n}\right)} - \frac{1}{\sin\left(\frac{t-\pi}{2n}\right)} \right] dt \\
&\leq 0,
\end{aligned}
$$

since the first factor of the integrand is non-negative and the second factor non-positive in the domain of integration. Thus we have shown that among the values $\{s_n(\frac{\pi(2k-1)}{2n}) : k = 1, \cdots, n\}$, the largest one is $s_n(\frac{\pi}{2n})$.

(d) Use the power series expansion of the sine function to show that

$$\sup_{s\in[0,\pi]} \left| \frac{1}{s} - \frac{1}{2n\sin(\frac{s}{2n})} \right| \to 0 \text{ as } n \to \infty.$$

This implies that

$$s_n\left(\frac{\pi}{2n}\right) = \frac{2}{\pi}\int_0^{\frac{\pi}{2n}} \frac{\sin 2nt}{\sin t}\, dt = \frac{2}{\pi}\int_0^{\pi} \frac{\sin s}{\sin\left(\frac{s}{2n}\right)} \frac{ds}{2n} \to \frac{2}{\pi}\int_0^{\pi} \frac{\sin s}{s}\, ds.$$

$\square$