Homework 2 Solutions - Math 321,Spring 2015

(1) For each $a \in [0,1]$ consider $f_a \in B[0,1]$ i.e. $f_a : [0,1] \to [0,1]$ defined by

$$f_a(t) = \begin{cases} 1 & \text{if } t = a \\ 0 & \text{if } t \neq a \end{cases}$$

There are uncountably many such f_a as $[0,1]$ is uncountable. Also if $a \neq b$ then $\|f_a - f_b\|_\infty = 1$ (if it is clear from context that we are using sup norm you may omit the ∞ symbol in the norm if you like.) Now consider all the balls $B(f_a, \frac{1}{2}), a \in [0,1]$ so if $f_b \neq f_a$ then $f_b \notin B(f_a, \frac{1}{2})$.
Now if $C \subseteq B[0,1]$ is dense then for each f_a there is a $g_a \in C$ such that $\|g_a - f_a\|_\infty < \frac{1}{2}$ and if $a \neq b$ then

$$\|f_b - g_a\| \geq \|f_b - f_a\| - \|f_a - g_a\| \geq 1 - \frac{1}{2} > 0$$

Hence for $a \neq b, g_a \neq g_b$ so the dense subset C cannot be countable.

(2) **Claim** If f_n is uniformly cauchy in $\mathbb{R}$ (or any metric space) then it is uniformly convergent in $\mathbb{R}$: Proof, for each x, $\{f_n(x)\}_{n\geq 1}$ is cauchy. So $f(x) := \lim_{n\to\infty} f_n(x)$ exists for every x. Now let $\epsilon > 0$ we can find $N, n, m > N$ implies $|f_m(x) - f_n(x)| < \epsilon$ for all x. Now with $n.m > N$ we have that for all x, we have

$$|f_n(x) - f(x)| \leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)| \leq \epsilon + |f(x) - f_m(x)|$$

Now let m goes to ∞ we have that if $n > N$ then $|f_n(x) - f(x)| < 2\epsilon$ for all x.(We only take pointwise limit in m, we choose m large enough so that $|f(x) - f_m(x)| < \epsilon$. , this m may be different for different x. Hence we have uniform convergence.
Now we will show that f_n is uniformly cauchy on $\mathbb{R}$. First since f_n, f_m are continuous, we can find δ_1, δ_2 such that for any $x \in \mathbb{R}$,

$$|y - x| < \delta_1 \implies |f_n(y) - f_n(x)| < \epsilon, |y - x| < \delta_2 \implies |f_m(y) - f_m(x)| < \epsilon.$$

Since $\mathbb{Q}$ is dense in $\mathbb{R}$. Let $\delta = \min \delta_1, \delta_2$, choose $x_r \in \mathbb{Q}$ (which may depend on x) such that $|x_r - x| < \delta$ Also since f_n is uniformly convergence on $\mathbb{Q}$, there is N such that if $n.m > N$ then

$$|f_n(x) - f_m(x)| < \frac{\epsilon}{3}$$

for all $x \in \mathbb{Q}$. Combine these information, we have that for all $n, m > N$,

$$|f_n(x) - f_m(x)| \leq |f_m(x) - f_n(x_r)| + |f_n(x_r) - f_m(x_r)| + |f_m(x_r) - f_m(x_r)| < 3\epsilon$$

Note You may argue that the limiting f is continuous on $\mathbb{R}$ and use it to do this . This is true but not completely obvious. You may have to argue why it is (e.g.

problem 7.11 in textbook) **Note** Important: sup norm of $f : X \to \mathbb{R}$ (or $\mathbb{C}$) is defined as

$$\|f\|_\infty = \sup_{x \in X} |f(x)|$$

(sometimes it is ∞ so is not defined). We should observe that this quantity depends only on f (and domain) , it does not depend on x so it is not necessary to write $\|f(x)\|_\infty$. The statement like $\|f_n(x) - f_n(y)\|_\infty < \epsilon$ or "$\|f_n(x) - f_m(x)\|_\infty < \epsilon$ for all x" is confusing and sometimes is a misuse of notation. If you don't like the notation you can also write in words e.g. you can either write $\|f_n - f_m\|_\infty < \epsilon$ or $|f_n(x) - f_n(x)| < \epsilon$ for all $x \in X$.

(3) 3a.1) $(f_n)_{n\in\mathbb{N}}$ is cauchy in $B(X)$ means it is uniformly cauchy (since we are using sup norm) Hence as in the proof of the claim in Q2, f converges uniformly to f. Now we have to check that $f \in B(X)$. To see this, f_N is bounded means there is an $M, |f_N(x)| \leq M$ for all x. Also there is and N such that $n > N \implies \|f_N - f\| < \epsilon$. Hence for all x,

$$|f(x)| \leq |f(x) - f_N(x)| + |f_N(x)| < M + \epsilon$$

so $f \in B(X)$.
3a.2) Now we prove that $\lim_{n\to\infty} \|f_n\|_\infty = \|f\|_\infty$ This follows from Triangle inequality:

$$|\|f_n\|_\infty - \|f\|_\infty| \leq \|f - f_n\|_\infty \to 0 (n \to \infty.)$$

And we have the result (or one can argue that $\|\|_\infty$ is continuous.
3a.3) To show that $\sup_n \|f_n\|_\infty < \infty$.(The face that each $f_n \in B(X)$ only tells us that $\|f_n\|_\infty$ exists as a finite number). First, there is $N, \forall n \geq N, \|f_n - f\|_\infty < 1$ i.e. $\|f_n\|_\infty \leq \|f\|_\infty + 1$ for all $n > N$. Let $M = \max\{\|f\|_\infty + 1, \|f_1\|_\infty, ..., , \|f_N\|_\infty\}$. Then $\|f_n\|_\infty \leq M$ for all n hence $\sup_n \|f_n\|_\infty \leq M < \infty$.

3b) Since $T_N = \sum_{n=1}^N \|g_n\|_\infty$ is cauchy. Let $\epsilon > 0$ there is an N such that for $n, m > N$, for all $x \in X$,

$$\Big| \sum_{k=n+1}^{m} g_k(x) \Big| \leq \sum_{k=n+1}^{m} \|g_k\|_\infty < \epsilon$$

This implies $S_N = \sum_{n=1}^N$ is a cauchy sequence in $B(X)$ (sup norm). Using Q3a) we conclude that S_N converges in $B(X)$. The triangle inequalities is true for finite sum but by Q 3a3) we can take $N \to \infty$ to get the required triangle inequality.

(4) 4a) $|\sum_{n\leq N} a_n \sin nx| \leq \sum_{n\leq N} |a_n|$ for all $x \in \mathbb{R}$ and $\sum_{n\leq N} |a_n| < \infty$. By Weierstrass M-test, we conclude that $\sum_{n\leq N} a_n \sin nx$ converges uniformly on $\mathbb{R}$. Same is true for $\sum_{n\leq N} a_n \cos nx$.

4b)If $|x| > 0$ then $1 + x^2 > 1$. Hence for $x \neq 0$ we can apply geometric series:

$$\sum_{n=1}^{\infty} \frac{x^2}{(1+x^2)^n} = \frac{x^2}{1+x^2} \sum_{n=0}^{\infty} \frac{1}{(1+x^2)^n} = \frac{x^2}{1+x^2} \frac{1}{1-\frac{1}{1+x^2}} = 1$$

While at $x = 0$ the series obviously converges to 0. The limiting function is not continuous at $x = 0$ so the convergence cannot be uniform.
Direct proof: $\lim_{x\to 0^+}(1+x^2)^N = 1$ so we can find $x \in (0, 1]$ such that $(1+x^2)^N \leq 2$ so with this x,

$$|\sum_{n\geq 1} \frac{x^2}{(1+x^2)^n} - \sum_{n\leq N} \frac{x^2}{(1+x^2)^n}| = |\sum_{n\geq N+1} \frac{x^2}{(1+x^2)^n}| = \frac{1}{(1+x^2)^N} \geq \frac{1}{2}$$

Hence the convergence can't be uniform.

(5) Write $x \in [0, 1]$ in base 3 expansion $x = 0.a_1a_2...$ then we know that $a_i = 0, 2$. Then it is not show that $f(3^n t) = \frac{a_n}{2}$. Now pick any $(x, y) \in [0, 1]^2$ whose base 2 expansion are $x = 0.a_0a_1a_2..., y = 0.b_0b_1b_2...$ Let $t \in [0, 1]$ has base 3 expansion $0.(2a_0)(2b_0)(2a_1)...$ then we can check that $t \in \Delta$ and $s(t) = (x, y)$. This shows $[0, 1]^2 \subseteq s(\Delta)$. On the other hand,

$$0 \leq \sum_{k=0}^{\infty} 2^{-k-1} f(3^{2k}t) \leq \sum_{k=0}^{\infty} 2^{-k-1} = 1$$

so $0 \leq x(t) \leq 1$ and the same is true for $y(t)$ hence $s(\Delta) \subseteq [0, 1]^2$.