Math 263 Assignment 6 Solutions

Problem 1. Find the volume of the solid bounded by the surfaces \( z = 3x^2 + 3y^2 \) and \( z = 4 - x^2 - y^2 \).

Solution. The two paraboloids intersect when \( 3x^2 + 3y^2 = 4 - x^2 - y^2 \) or \( x^2 + y^2 = 1 \). Writing down the given volume first in Cartesian coordinates and then converting into polar form we find that

\[
V = \int \int_{x^2+y^2 \leq 1} [(4 - x^2 - y^2) - (3x^2 + 3y^2)] \, dA
\]

\[
= \int_0^{2\pi} \int_0^1 (4 - r^2) r \, dr \, d\theta
\]

\[
= \int_0^{2\pi} (4r - 4r^3) \, dr = 2\pi.
\]

Problem 2. Sketch the region enclosed by the curve \( r = b + a \cos \theta \) and compute its area. Here \( a \) and \( b \) are positive constants, \( b > a \).

Solution. The curve is a cardioid symmetric about the \( x \)-axis. The area enclosed by it is

\[
A = 2 \int_0^{\pi} \int_0^{b + a \cos \theta} r \, dr \, d\theta
\]

\[
= \int_0^{\pi} \left( b + a \cos \theta \right)^2 \, d\theta
\]

\[
= \int_0^{\pi} \left[ b^2 + a^2 \frac{1}{2} \left( 1 + \cos(2\theta) \right) + 2ab \cos \theta \right] \, d\theta
\]

\[
= \left( b^2 + \frac{a^2}{2} \right) \pi.
\]

Problem 3. A lamina occupies the region inside the circle \( x^2 + y^2 = 2y \) but outside the circle \( x^2 + y^2 = 1 \). Find the center of mass if the density at any point is inversely proportional to its distance from the origin.

Solution. The circles \( x^2 + y^2 = 2y \) and \( x^2 + y^2 = 1 \) may be written in polar coordinates as \( r = 2 \sin \theta \) and \( r = 1 \) respectively. They intersect at two points, where \( \sin \theta = \frac{1}{2} \), so that \( \theta = \frac{\pi}{6} \) and \( \theta = \frac{5\pi}{6} \). Further the density function is \( \rho(x, y) = k/\sqrt{x^2 + y^2} = k/r \),
where \( k \) is the constant of proportionality. Therefore

\[
\text{mass } m = \int \frac{5 \pi}{6} \int_1^{2 \sin \theta} \frac{k}{r} r dr \ d\theta \\
= k \int \frac{5 \pi}{6} (2 \sin \theta - 1) d\theta \\
= 2k(\sqrt{3} - \frac{\pi}{3})
\]

By symmetry of the domains and the function \( f(x) = x \), we know that \( M_y = 0 \), and

\[
M_x = \int \frac{5 \pi}{6} \int_1^{2 \sin \theta} kr \sin \theta dr \ d\theta \\
= \frac{k}{2} \int \frac{5 \pi}{6} (4 \sin^3 \theta - \sin \theta) d\theta \\
= \sqrt{3}k.
\]

Hence \((\bar{x}, \bar{y}) = (0, \frac{3\sqrt{3}}{2(3\sqrt{3} - \pi)})\). \( \square \)

**Problem 4.** Evaluate the triple integral

\[
\iiint_E z dV,
\]

where \( E \) is bounded by the cylinder \( y^2 + z^2 = 9 \) and the planes \( x = 0, y = 3x \) and \( z = 0 \) in the first octant.

**Solution.**

\[
\iiint_E z dV = \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} z \ dz \ dy \ dx \\
= \int_0^1 \int_{3x}^3 \frac{1}{2} (9 - y^2) \ dy \ dx \\
= \int_0^1 \left[ \frac{9y}{2} - \frac{y^3}{6} \right]_{y=3x}^{y=3} \ dx \\
= \int_0^1 \left[ 9 - \frac{27}{2} x + \frac{9}{2} x^3 \right] dx = \frac{27}{8}.
\]

**Problem 5.** Find the volume of the solid bounded by the cylinder \( y = x^2 \) and the planes \( z = 0, z = 4 \) and \( y = 9 \).
Solution.

\[ V = \iiint_E dV = \int_{-3}^{3} \int_{x^2}^{9} \int_0^4 dz \, dy \, dx \]
\[ = 4 \int_{-3}^{3} \int_{x^2}^{9} dy \, dx \]
\[ = 4 \int_{-3}^{3} (9 - x^2) \, dx \]
\[ = 144. \]

\[ \square \]

Problem 6. Sketch the solid whose volume is given by the iterated integral

\[ \int_0^2 \int_0^{2-y} \int_0^{1-y^2} dx \, dz \, dy. \]

Solution. The triple integral is the volume of \( E = \{(x, y, z) : 0 \leq y \leq 2, 0 \leq z \leq 2 - y, 0 \leq x \leq 4 - y^2\} \), the solid bounded by the three coordinate planes, the plane \( z = 2 - y \), and the cylindrical surface \( x = 4 - y^2 \).

\[ \square \]

Problem 7. Rewrite the integral

\[ \int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x, y, z) \, dy \, dz \, dx \]

as an equivalent iterated integral in five other orders.

Solution. The projection of \( E \) onto the \( xy \) plane is the right triangle bounded by the coordinate axes and the straight line \( x + y = 1 \). On the other hand, the projection onto the \( xz \) plane is the region bounded by the coordinate axes and the parabola \( z = 1 - x^2 \). Therefore the given iterated integral may also be written as

\[ \int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x, y, z) \, dy \, dz \, dx \]
\[ = \int_0^1 \int_0^{\sqrt{1-x}} \int_0^{1-x} f(x, y, z) \, dy \, dx \, dz \]
\[ = \int_0^1 \int_0^{1-x} \int_0^{1-x^2} f(x, y, z) \, dz \, dx \, dy \]
\[ = \int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x, y, z) \, dz \, dy \, dx. \]

Now the surface \( z = 1 - x^2 \) intersects the plane \( y = 1 - x \) in a curve whose projection in the \( yz \)-plane is \( z = 1 - (1 - y)^2 \) or \( z = 2y - y^2 \). So we must split up the projection of \( E \) on
the $yz$ plane (which is the unit square) into two regions, whose boundary is the curve above. The given integral is therefore also equal to

\[
\int_0^1 \int_0^{1-\sqrt{1-z}} \int_0^{\sqrt{1-z}} f(x, y, z) \, dx \, dy \, dz + \int_0^1 \int_{1-\sqrt{1-z}}^1 \int_0^{1-y} f(x, y, z) \, dx \, dy \, dz
\]

\[
= \int_0^1 \int_0^{2y-y^2} \int_0^{1-y} f(x, y, z) \, dx \, dy \, dz + \int_0^1 \int_0^{1-\sqrt{1-z}} \int_0^{\sqrt{1-z}} f(x, y, z) \, dx \, dy \, dz.
\]