Q1 [10 marks]

The critical points of $f(x, y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$ are $(0, 0), (2, 0), (1, 1)$, and $(1, -1)$. Use the second derivative test to classify these critical points.

$f_{xx}(x,y) = 6x - 6$

$f_{yy}(x,y) = 6x - 6$

$f_{xy}(x,y) = 6y$

$\Rightarrow D(x,y) = f_{xx} f_{yy} - f_{xy}^2 = (6x-6)^2 - (6y)^2$

$(0,0)$: $\left.\begin{array}{l} D(0,0) = 36 > 0 \\ f_{xx}(0,0) = -6 < 0 \end{array}\right\} \Rightarrow (0,0)$ is a local max.

$(2,0)$: $\left.\begin{array}{l} D(2,0) = 36 > 0 \\ f_{xx}(2,0) = 6 > \end{array}\right\} \Rightarrow (2,0)$ is a local min.

$(1,1)$: $D(1,1) = -36 < 0 \Rightarrow (1,1)$ is a saddle pt.

$(1,-1)$: $D(1,-1) = -36 < 0 \Rightarrow (+1,-1)$ is a saddle pt.

Q2 [10 marks]

Use Lagrange multipliers to find the maximum and minimum values of $f(x, y) = x^2y$ subject to the constraint $x^2 + 2y^2 = 6$.

$$\mathcal{L}(x,y,\lambda) = x^2y + \lambda(x^2+2y^2-6).$$

① $\frac{\partial \mathcal{L}}{\partial x} = 2xy + 2\lambda x = 0$; ② $\frac{\partial \mathcal{L}}{\partial y} = x^2 + 4\lambda y = 0$; ③ $\frac{\partial \mathcal{L}}{\partial \lambda} = x^2+2y^2$

① $\Rightarrow x(2y+2\lambda)=0$. Either $x=0$ OR $\lambda = -y$.

If $x=0$ then ③ says $x^2+2y^2-6=0 \Rightarrow \boxed{y = \pm\sqrt{3}}$.

If $x \neq 0$ then $\lambda = -y$ so, ② says

$$x^2+4(-y)y)=0 \Rightarrow x^2-4y^2=0$$

$$\Rightarrow x^2 = 4y^2.$$

Plug this into ③ to get $4y^2+2y^2-6=0$

$$\Rightarrow 6y^2-6=0 \Rightarrow \boxed{y=\pm 1.}$$

$$\Rightarrow \boxed{x=\pm 2}$$

Possible max/min: $(0,\sqrt{3})$, $(0,-\sqrt{3})$, $(2,1)$, $(2,-1)$, $(-2,1)$, $(-2,-1)$.

$f(0,\sqrt{3}) = f(0,-\sqrt{3}) = 0$

$f(2,1) = f(-2,1) = 4$ ← max at $(2,1)$ and $(-2,1)$

$f(2,-1) = f(-2,-1) = -4$. ← min at $(2,-1)$ and $(-2,-1)$.

Q3 [10 marks]

Buy Here corner store carries two brands of frozen orange juice. Brand A costs 30 cents per can, while brand B costs 40 cents per can. The store manager estimates that if brand A sells for x cents per can and brand B sells for y cents per can, then she will sell $70 - 5x + 4y$ cans of brand A and $80 + 6x - 7y$ cans of brand B each day. How should the store manager price each brand to maximize the profit from the sale of frozen orange juice? Remember, profit equals revenue minus cost.

Note: Profit will be generated by the difference between selling price and cost per can.

$$\text{Profit} = P(x,y) = \underbrace{(70-5x+4y)}_{\#\text{ cans A}}\underbrace{(x-30)}_{\text{profit per can.}} + \underbrace{(80+6x-7y)}_{\#\text{ cans B}}\underbrace{(y-4}_{\text{profit per can}}$$

$$\Rightarrow P(x,y) = -5x^2 + 10xy - 20x - 7y^2 + 240y - 5300$$

$$\Rightarrow \left.\begin{aligned} \frac{\partial P}{\partial x} &= -10x + 10y - 20 = 0 \\ \frac{\partial P}{\partial y} &= 10x - 14y + 240 = 0 \end{aligned}\right\} \quad x = 53, \; y = 55.$$

Check it is a max: $P_{xx}(x,y) = -10$, $P_{yy}(x,y) = -14$, $P_{xy}(x,y) = 10$

$$\left.\begin{aligned} D(53,55) &= (-10)(-14) - (10)^2 > 0 \\ P_{xx}(53,55) &< 0 \end{aligned}\right\} \Rightarrow (53,55) \text{ is a max}$$

So sell brand A at 53¢ per can & brand B at 55¢ per ca