Q2 [15 marks]

A manufacturer is planning to sell a new product at \$150 per unit, and estimates that if it spends x thousand dollars on development and y thousand dollars on promotion, then it will sell

$$q = q(x, y) = \frac{320y}{y+2} + \frac{160x}{x+4}$$

units of product. The cost of manufacturing the product is \$50 per unit. The manufacturer has a total of \$8000 to spend on development and promotion. Use the method of Lagrange multipliers to find how this money should be allocated to generate the largest possible **profit**. [*Hint:* Profit = (number of units sold)(price per unit - cost per unit)- (amount spent on development and promotion).]

$$\text{Profit} = P(x,y) = q(x,y)(150-50) - \overset{8\cancel{000}}{\overset{\shortparallel}{(x+y)}}1000$$

$$= 100\left(\frac{320y}{y+2} + \frac{160x}{x+4}\right) - \cancel{(x+y)}\ \cancel{8000}\ 8000$$

3

$$\max\ P(x,y) \text{ subject to } x+y = 8\cancel{000}.$$

since x, y in thousands

$$\mathcal{L}(x,y,\lambda) = P(x,y) + \lambda(x+y-8\cancel{000})$$

$$= 100\left(\frac{320y}{y+2} + \frac{160x}{x+4}\right) - \overset{8000}{\cancel{x+y}}\ \cancel{8000} + \lambda(x+y-8\cancel{000})$$

4

(QUESTION 2 CONTINUED)

$$\text{①}\ \frac{\partial \mathcal{L}}{\partial x} = \frac{\cancel{64}000}{(x+4)^2} + \lambda = 0$$

$$\text{②}\ \frac{\partial \mathcal{L}}{\partial y} = \frac{64000}{(y+2)^2} + \lambda = 0$$

$$\text{③}\ \frac{\partial \mathcal{L}}{\partial x} = x + y - 8 = 0$$

3

① = ② $\frac{1}{(x+4)^2} = \frac{1}{(y+2)^2}$ 3 $\Rightarrow (x+4)^2 = (y+2)^2$

$\Rightarrow x^2 + 8x + 16 = y^2 + 4y + 4$

$y = 8 - x$

$x^2 + 8x + 16 = (8-x)^2 + 4(8-x) + 4$

$\Rightarrow \cancel{x^2} + 8x + \cancel{16} = 64 - 16x \cancel{+x^2} + 32 - 4x + 4$

$\Rightarrow 28x = 90 - 16$

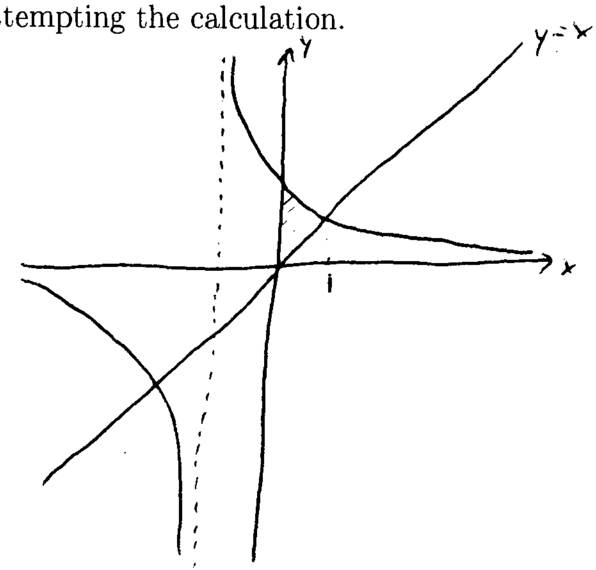
$\Rightarrow 28x = 74$

$\boxed{x = 3}$

$\Rightarrow \boxed{y = 5}$ 2

Q3 [5 marks]

Find the area of the finite region bounded by the y-axis, the curve $y = x$, and the curve $y = \frac{2}{x+1}$. It will be useful to sketch the region before attempting the calculation.



Find intercepts:

$$\frac{2}{x+1} = x$$

$\Rightarrow 2 = (x+1)x$

$\Rightarrow 0 = (x+1)x - 2$

$\Rightarrow 0 = x^2 + x - 2$

$\Rightarrow 0 = (x+2)(x-1)$

$\Rightarrow x = -2$ or $x = 1$

$\uparrow$

reject this because the y-axis is a bound; see diagram. 2 marks

For $0 \le x \le 1$, $\frac{2}{x+1} \ge x$, so the area is given by:

$$\int_0^1 \frac{2}{x+1} - x \, dx = 2\ln|x+1| - \frac{x^2}{2}\Big|_0^1 = 2\ln(2) - \frac{1}{2} - 2\ln(1)$$

$$= 2\ln(2) - \frac{1}{2}$$

3 marks

Q4 [5 marks]

Suppose that the marginal revenue function for a company producing q units of a product is

$$MR(q) = R'(q) = 400 - 3q^2.$$

Find the *additional* revenue received from doubling production if currently 10 units are being produced.

Additional Revenue

$$= \int_{10}^{20} R'(q)\, dq$$

$$= \int_{10}^{20} 400 - 3q^2 \, dq$$

$$= 400q - q^3 \Big|_{10}^{20}$$

$$= 8000 - 8000 - 4000 + 1000$$

$$= -3000.$$

Q5 [10 marks]

A tire manufacturer estimates that q thousand radial tires will be demanded by wholesalers when the price is

$$p = D(q) = -0.1q^2 + 90$$

dollars per tire, and that the same number of tires will be supplied when the price is

$$p = S(q) = 0.2q^2 + q + 50$$

dollars per tire.

(a) Find the equilibrium price (i.e. where the supply and demand curves intersect) and the quantity supplied and demanded at that price. A sketch may be useful.

$D(q) = S(q)$

$\Rightarrow -0.1q^2 + 90 = 0.2q^2 + q + 50$ 1 mark

$\Rightarrow 0 = 0.3q^2 + q - 40$

$\Rightarrow 0 = (0.3q + 4)(q - 10)$

$\Rightarrow q = \frac{-4}{0.3}$ or $\boxed{q = 10}$ 3 marks

reject this, q must be positive

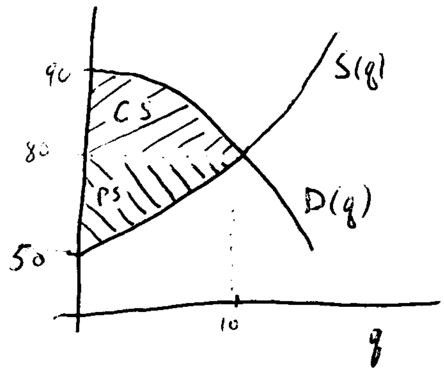
$p = D(q)$

$p = D(10) = -0.1(10)^2 + 90 = 80$

1 mark

$p = 80$

(*b*) Determine the consumers' and the producers' surpluses at the equilibrium price.



$$CS = \int_0^{10} D(q) - 80 \, dq$$

$$= \int_0^{10} -0.1q^2 + 90 - 80 \, dq$$

$$= \int_0^{10} -0.1q^2 + 10 \, dq$$

$$= -0.1 \cdot \frac{q^3}{3} + 10q \Big|_0^{10}$$

$$= -\frac{(10)^3}{30} + 100$$

$$= \frac{200}{3}$$

$$PS = \int_0^{10} 80 - S(q) \, dq$$

$$= \int_0^{10} 80 - (0.2q^2 + q + 50) \, dq$$

$$= \int_0^{10} -0.2q^2 - q + 30 \, dq$$

$$= -0.2 \frac{q^3}{3} - \frac{q^2}{2} + 30q \Big|_0^{10}$$

$$= -\frac{2(10)^3}{30} - \frac{(10)^2}{2} + 300$$

$$= -\frac{200}{3} - \frac{100}{2} + 300$$

$$= \frac{550}{3}$$

3 marks: correct set-up

2 marks: integration

Q6 [5 marks]

An oil well that yields 900 barrels of crude oil per month will run dry in 3 years. The price of crude oil is currently \$40 per barrel. If the oil is sold as soon as it is extracted from the ground and the money invested at 5% per year compounded continuously, what will be the total future value of the revenue from the well over its 3-year lifetime?

900 barrels/month × 12 months/year

= 10 800 barrels/year.

Income stream → \$40 per barrel × 10 800 barrels per year.

\$432 000 per year.

@ 5% per annum.
cont. compounding.

~~$I(t)t$~~

$$dI(t) = 432\,000\, e^{(3-t)0.05}\, dt$$

$$\text{Revenue} = \int_0^3 432\,000\, e^{(3-t)0.05}\, dt$$

(compute numerical answer)