1. \[ \lim_{x \to 2} \frac{x^2 - 6x + 8}{x^2 - 4} \]

\[ \frac{0}{0} \]?

**Technique:** Factor.

\[ x^2 - 6x + 8 = (x - 2)(x - 4) \]
\[ x^2 - 4 = (x + 2)(x - 2) \]

\[ \lim_{x \to 2} \frac{(x - 2)(x - 4)}{(x + 2)(x - 2)} = \lim_{x \to 2} \frac{x - 4}{x + 2} = -\frac{1}{2} \]

2. \[ \lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9} \]

\[ \frac{0}{0} \]?

*Multiply with* \( \sqrt{x} + 3 \) \( \text{algebraic conjugate of} \sqrt{x} - 3 \)

\[ \frac{(\sqrt{x} - 3)(\sqrt{x} + 3)}{(x - 9)(\sqrt{x} + 3)} = \frac{(\sqrt{x})^2 - 3^2}{(x - 9)(\sqrt{x} + 3)} = \frac{x - 9}{x - 9} \cdot \frac{\sqrt{x} + 3}{\sqrt{x} + 3} = 1 \]

\[ \lim_{x \to 9} \frac{1}{3 + 3} = \frac{1}{6} \]
\[
\lim_{x \to 9} \frac{\sqrt{x} - 3}{\sqrt{x} - 9} = \frac{3 - 9}{3 - 9} = -6
\]

\[
\lim_{x \to 9} \frac{\sqrt{x} - 9}{\sqrt{x} - 3} = \frac{6}{3} = 2
\]

\[
\lim_{x \to 9} \frac{\sqrt{x} - 9}{\sqrt{x} + 3} = \text{does not exist.}
\]

\[
\text{Ex. } f(x) = \begin{cases} x^2 + 1 & \text{if } x > -1 \\ \frac{1}{x+1} & \text{if } x = -1 \end{cases}
\]

\[
\lim_{x \to -1^-} f(x)? \quad \lim_{x \to -1^+} f(x)? \quad \lim_{x \to -1} f(x)? \quad \lim_{x \to 1^-} f(x)? \quad \lim_{x \to 1^+} f(x)? \quad \lim_{x \to 1} f(x)?
\]

\[
f(-1) = \sqrt{-1+1} = 0
\]

\[
\lim_{x \to -1^-} f(x) = (-1)^2 + 1 = 2
\]

\[
\lim_{x \to -1^+} f(x) = \sqrt{-1+1} = 0
\]

\[
\lim_{x \to 1^-} f(x) \text{ does not exist because } \lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x).
\]
Continuity at Point

\[ f(x) \]

\[ a \quad \rightarrow \quad x \]

\( f(x) \) is continuous at \( x = a \) if we can draw \( f(x) \) without lifting the pen.

**Definition.**
A function \( f \) is continuous at \( x = a \) if \( \lim_{{x \to a}} f(x) = f(a) \).

If \( f \) is not continuous at \( a \), then \( a \) is a point of discontinuity.

\[ \lim_{{x \to a}} f(x) \]

DNE

\( f(x) \) is not continuous at \( x = a \)
check list

1. \( f(a) \) is defined. \( f(a) \rightarrow v \)

2. \( \lim_{x \to a} f(x) \) exist

3. \( \lim_{x \to a} f(x) = f(a) \)

points of discontinuity?

\( x=1 \). \( f(1) \) is not defined.

\( x=5 \). \( f(5) \rightarrow v \) \( f(5) \) not defined.

\( x=2 \)

\( \lim_{x \to 2^-} f(2) = 1 \)
\( \lim_{x \to 2^+} f(2) = 3 \)
\( \lim_{x \to 2} f(2) \neq 1 \)
\( \lim_{x \to 2} f(2) \neq 3 \)

\( x=3 \).

\( \lim_{x \to 3^+} f(3) = 1 \)
\( \lim_{x \to 3^-} f(3) = 2 \)
\( \lim_{x \to 3} f(3) \neq 1 \)
\( \lim_{x \to 3} f(3) \neq 2 \)
\( \lim_{x \to 3} f(3) \) DNE (Does Not Exist)
Continuity Rule.

If $f$ and $g$ are continuous at $a$, then the following functions are also continuous at $a$.

a. $f + g$

b. $f - g$

c. $cf$

d. $fg$

e. $\frac{f}{g}$, $g(a) \neq 0$

f. $(f(x))^n$.

Why? We know $\lim_{x \to a} f(x) = f(a)$, $\lim_{x \to a} g(x) = g(a)$.

Want: $f + g$ is continuous at $a$.

To say: I need $\lim_{x \to a} (f(x) + g(x)) = f(a) + g(a)$.

Because $\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = f(a) + g(a)$.
Theorem 2.10.

(a) polynomial function.
   is continuous at all \( x \).

(b) rational function of
   the form \( \frac{P(x)}{Q(x)} \).
   continuous at all \( x \), \( Q(x) \neq 0 \).

Ex. \( f(x) = \frac{x^5 + 6x + 17}{x^2 - 9} \):

for what values \( x \), \( f(x) \) continuous.

\( x^2 - 9 \neq 0 \), \( x \neq -3, 3 \).

\( f(x) \) is continuous at all \( x \)
except \( -3, 3 \).
\[ f(g(x)) = \left( \frac{x^4 - 2x + 2}{x^6 + 2x^4 + 1} \right)^{10} \]

\[ g(x) \text{ continuous at } 0 \Rightarrow \]
\[ g(x) \text{ is rational function.} \]

\[ 0^6 + 20^4 + 1 = 1 \Rightarrow \]
\[ f(g(0)) = f(0) \text{ continuous at } x = 0 \]
\[ g(0) = \frac{2}{1} = 2. \]

\[ f(x) \text{ is continuous at } 2. \]

\[ \lim_{x \to 0} f(g(x)) = f(g(0)) = f(2) = 2^0. \]

Answer: 2^0.
Theorem 2.12

\[ \lim_{x \to a} f(g(x)) = f(g(a)) \]

(1) If \( g \) is continuous at \( a \),
\( f \) is continuous at \( g(a) \),
Then:
\[ \lim_{x \to a} f(g(x)) = f(g(a)) \]

Ex. \[ \lim_{x \to 0} \left( \frac{x^4 - 2x + 2}{x^6 + 2x^4 + 1} \right)^{10} \]

\( f(x) = \frac{x^4 - 2x + 2}{x^6 + 2x^4 + 1} \)
\( f(x) = x^{10} \)
(2) If $\lim_{x \to a} g(x) = L$ and $f$ continuous at $L$, then $\lim_{x \to a} f(g(x)) = f(L)$. $g(a)$ may not exist.