Instantaneous velocity

Ex. Suppose that a rock launched vertically upward from the ground with speed 96 ft/s.

From physics, the position of the rock after $t$ seconds given by $S(t) = -16t^2 + 96t$.

Q: What's the instantaneous velocity at $t = 1$ s?

$t = 1$ s \quad S(1) = 80 \text{ ft}$

$t = 2$ s \quad S(2) = 128 \text{ ft}$

$t = 3$ s \quad S(3) = 144 \text{ ft}$

\ldots
average velocity at \([1, 2]\)

\[ V_{av} = \frac{128 - 50}{1} = 48 \text{ ft/s} \]

\([1, t]\)

\[ V_{avg} = \frac{S(t) - S(1)}{t - 1} = \frac{S(t) - 80}{t - 1} \]

\[ V_{int} = \lim_{t \to 1} V_{avg} = \lim_{t \to 1} \frac{S(t) - 80}{t - 1} \]
<table>
<thead>
<tr>
<th>to</th>
<th>t_i</th>
<th>\Delta t</th>
<th>V_{avg} (ft/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
<td>48</td>
</tr>
<tr>
<td>1</td>
<td>1.5</td>
<td>0.5</td>
<td>56</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0.1</td>
<td>62.4</td>
</tr>
<tr>
<td>1</td>
<td>1.001</td>
<td>0.001</td>
<td>63.84</td>
</tr>
<tr>
<td>1</td>
<td>1.001</td>
<td>0.001</td>
<td>63.984</td>
</tr>
</tbody>
</table>

\[ V_{ins} = \lim_{t \to 1} V_{avg} = 64 \text{ ft/s} \]
\[ V_{\text{avg}} = \text{slope of the secant line.} \]

\[ \text{tangent line} \]

\[ \text{tangent line} \]

\[ V_{\text{int}} = \text{slope of tangent line.} \]

\[ m_{\text{tan}} = \lim_{t \to 1} \frac{S(t) - S(1)}{t - 1} \]
Definition

Limit of a function.

Suppose the function \( f \) is defined for all \( x \) near \( a \) except possibly at \( a \). If \( f(x) \) is arbitrarily close to \( a \), for all \( x \) sufficiently close to \( a \), we write \( \lim_{x \to a} f(x) = L \).

and say the limit of \( f(x) \) approaches a equal \( L \).

Remarks: \( \lim_{x \to a} f(x) \) not necessarily equal to \( f(a) \) can be different than \( f(a) \).
\[ \lim_{x \to 2} f(x) = 3. \]
\[ f(2) = 5 \]

For \( f(x) \):

For \( g(x) \):

\[ g(0) = 1 \]
\[ g(1) = 2 \]
\[ \lim_{x \to 1} g(x) = 2 \]
One sided limit

1. Right sided limit.

Suppose $f$ is defined for all $x$ near $a$, with $x > a$. If $f(x)$ is arbitrarily close to $L$ for all $x$ sufficiently close with $x > a$, we write:

$$\lim_{{x \to a^+}} f(x) = L$$
2. Left sided limit.

... if \( f(x) \) is arbitrarily close to \( L \) for \( x \) sufficiently close with \( x < a \)

\[
\lim_{{x \to a^-}} f(x) = L
\]

**Theorem 2.1**

Assume \( f(x) \) is defined for all \( x \) near \( a \) except possibly at \( a \). Then \( \lim_{{x \to a}} f(x) = L \) if and only if \( \lim_{{x \to a^+}} f(x) = L \) and \( \lim_{{x \to a^-}} f(x) = L \).
a. \( \lim_{x \to 2^-} g(x) = 4 \)

b. \( \lim_{x \to 2^+} g(x) = 1 \)

c. \( \lim_{x \to 2} g(x) = \text{does not exist. by theorem.} \)
Example

\[
\lim_{x \to 0} \cos \left( \frac{1}{x} \right) \quad ?
\]

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \cos \left( \frac{1}{x} \right) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>0.56238</td>
</tr>
<tr>
<td>0.0001</td>
<td>-0.95216</td>
</tr>
<tr>
<td>0.00001</td>
<td>-0.99996</td>
</tr>
<tr>
<td>0.000001</td>
<td>-0.36330</td>
</tr>
</tbody>
</table>

let \( x = \frac{1}{n\pi} \) \ n positive integer

\[
\cos \left( \frac{1}{x} \right) = \cos \left( \frac{1}{n\pi} \right) = \begin{cases} 1 & \text{if } n \text{ even} \\ -1 & \text{if } n \text{ odd} \end{cases}
\]

when \( n \) increases \( x \to 0 \)

\( \cos \left( \frac{1}{x} \right) \) oscillate between \( -1, 1 \)
\[ \lim_{x \to 0} \cos x \quad \text{DO NOT EXIST.} \]