Thm 4.7. Second Derivative test for local extrema.

1. If \( f'(c) = 0 \)
2. If \( f''(c) > 0 \), then \( f \) has a local minimum at \( x = c \).
3. If \( f''(c) < 0 \), then \( f \) has a local maximum at \( x = c \).
4. If \( f''(c) = 0 \), then \( f \) is inconclusive.

\[ \begin{array}{c}
\text{local min} \\
\vdots
\end{array} \]

\[ \text{local max} \]

\[ f'(c) > 0 \]

\[ f'(c) < 0 \]
Ex. 2. Apply

Goal: √ First derivative test.
& second derivative test.
to find local minimum
and local maximum.

Recap

\[ f(x) \]
\[ x \to \quad y \to \]

increasing function.

\[ f''(x) > 0 \]

concave down.

\[ f''(x) < 0 \]

concave up function.

increasing function.
Ex. \[ f(x) = \frac{x^4}{4} - \frac{5x^3}{3} - 4x^2 + 48x \]

Find local extrema.

1st Step: Find critical points. \( f'(x) = 0 \)

\[ f'(x) = x^3 - 5x^2 - 8x + 48 \cdot x = 4 \text{ is zero.} \]

\[
\begin{align*}
\text{Critical point. } x &= 4, x = -3 \\
\end{align*}
\]

\[ f''(x) = 3x^2 + 30 - 8 = 49 > 0 \]

\[ f''(4) = 3 \cdot 4^2 - 40 - 8 = 0 \]
\( f''(x) = 3x^2 - 10x - 8 \)

By 2nd derivative test (Thm 4.7)
\( x = \frac{5}{3} \) is local minimum

\[
\frac{1}{4} \quad 5
\]

\( f'''(x) = 0 \)
\( f'(5) > 0 \)

By 1st derivative test
\( x = 4 \) is neither local maximum nor local minimum
Ex. \( f(x) = \sin^2 x = (\sin x)^2 \neq \sin x^2 \)

Locate local extrema using 2nd derivative test.

Sol: \( f'(x) = 2 \sin x \cos x = \sin 2x \)

\[ f'(x) = 2 \cos^2 x - 2 \sin x \]

Solve for \( f'(x) = 0 \)

\[ \Rightarrow \sin 2x = 0 \]

\[ \Rightarrow 2x = n\pi \quad n = 0, 1, 2, \ldots \]

Critical points: \( x = \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \ldots \)

\[ f''(0) = 2 > 0 \quad \text{local min.} \]

\[ f''\left(\frac{\pi}{2}\right) = -2 < 0 \quad \text{local max.} \]

\[ f''(\pi) = 2 > 0 \quad \text{local min.} \]
Graph:

local max and local min alternating.

local min at $x = k\pi$, $k = \pm 0, 1, 2, \ldots$

local max at $x = \frac{(2k+1)\pi}{2}$, $k = \pm 0, 1, 2, \ldots$
Q: Does reflection point $x=c$ indicate $f''(c)=0$?

No. Counter example

$x=c$ is an inflection point. $f''(c)$ does not exist because it has a vertical tangent line.
Theorem 4.5.

Suppose $f$ continuous on $I$. Suppose $I$ contains one local extremum at $c$.

- If a local maximum occurs at $c$, then $f(c)$ is the absolute maximum of $f$ on $I$.
- ... a local minimum ...
  
  ..., ........ is the absolute minimum of $f$ on $I$.

[Diagrams]

Read: A Example 5.