§ 9.1 Taylor Polynomial.

Review Polynomial.

Degree 0: \( c_0 \).

Degree 1: \( c_1 x + c_0 \).

Degree 2: \( c_2 x^2 + c_1 x + c_0 \).

Degree n: \( c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 \).

Power series: \( \sum_{n=0}^{\infty} c_n x^n \). \( \sum_{n=0}^{\infty} C_n x^n \).

Linear approximation to \( f(x) \) at \( x = a \).

\( y = f(a) + f'(a) (x-a) \) \( \leftrightarrow \) First degree polynomial.

\( P_1(x) = f(a) + f'(a) (x-a) \).

To have better approximate to \( f(x) \) near \( a \), we use quadratic approximating polynomial (degree 2 polynomial).

\( P_2(x) \).
\[ P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 \]

**Goal:** To find \( C_2 \).

In order to have better approximation

\[ P_2(a) = f(a) \quad \checkmark \]
\[ P_2'(a) = f'(a) \quad \checkmark \]
\[ P_2''(a) = f''(a) \quad \checkmark \]

\[ P_2(x) = 2C_2 \]
\[ 2C_2 = f''(a) \]

\[ C_2 = \frac{f''(a)}{2} \]

The quadratic approximating polynomial to \( f(x) \) at \( x = a \)

\[ P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2. \]
Ex. $f(x) = \ln x$. at $x = 1$.

Linear approximation. at $x = 1$. (Given)

$$P(x) = f(1) + f'(1) (x-1)$$

$$= 0 + 1 (x-1)$$

$= x - 1$.

Quadratic approximation

$$P_2(x) = f(1) + f'(1) (x-1) + \frac{f''(1)}{2!} (x-1)^2$$

$$= 0 + (x-1) + -\frac{1}{2} (x-1)^2$$

To approximate $\ln (1.05)$

$$P_1(1.05) = 0.05.$$

$$P_2(1.05) = 0.05 - \frac{1}{2} (0.05)^2$$

$= 0.04875.$

Exact value in calculator $\ln 1.05 = 0.04879$
Assume $f$ and its first $n$-th derivative exist at $x=a$.

To find an $n$-th polynomial to estimate $f(x)$ near $a$.

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$n! = n \cdot (n-1) \cdot (n-2) \cdots 1$

$P_n(x)$ satisfying:

$$P_n(a) = f(a)$$
$$P_n'(a) = f'(a)$$
$$\vdots$$
$$P_n^{(n)}(a) = f^{(n)}(a)$$
Taylor Polynomial.

The n-th order Taylor polynomial for $f$, with center at $a$, is:

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

or, write it as:

$$P_n(x) = \sum_{k=0}^{n} C_k (x-a)^k \quad C_k = \frac{f^{(k)}(a)}{k!}, \quad k=0,1,2,\ldots,n$$

Ex. Find the Taylor polynomial of $P(x), P'(x), \ldots P^{(n)}(x)$ centered at $x=0$ for $f(x) = \sin x$. 