1.11 Numerical Integration: Supplemental Examples

Ex. 1: Given an upper bound for the error in approximating
\[ \int_{-2}^{2} \frac{1}{12} x^4 - 2x^2 \, dx \]
by using the Trapezoidal Rule with 20 subintervals. Use the formula:
\[ \left| \int_{a}^{b} f(x) \, dx - T_N \right| \leq \frac{M(b-a)^3}{12N^2} \]
where M is a number such that \( |f''(x)| \leq M \) for all \( a \leq x \leq b \).

Solution: For us, \( a = -2, b = 2, \) and \( N = 20 \). It remains to find a suitable M. First, note that
\[ f''(x) = x^2 - 4. \]

On the interval \([-2, 2]\), \( f''(x) \) achieves a minimum value of \(-4\) and a maximum value of 0; therefore, \( |f''(x)| \leq 4 \) for all \(-2 \leq x \leq 2\). So we take \( M = 4 \), and apply the formula to see that
\[ \left| \int_{-2}^{2} \frac{1}{12} x^4 - 2x^2 \, dx - T_{20} \right| \leq \frac{4 \cdot 4^3}{12 (20)^2}. \]

Note also that for all \( x \) in \([-2, 2]\), \( f''(x) \leq 0 \), and so \( f(x) \) is concave down on this interval. This means that \( T_{20} \) is an underestimate for the true value of \( \int_{-2}^{2} \frac{1}{12} x^4 - 2x^2 \, dx \).
Ex. 2: Give an upper bound for the error in approximating 
\[ \int_{-2}^{-1} 3 \times x^{-\frac{1}{3}} \, dx \] by using the Midpoint Rule with 10 subintervals. Use the formula: \[ |\int_a^b f(x) \, dx - M_N| \leq \frac{M(b-a)^3}{24 \, N^2}, \] where M is a number such that \( |f''(x)| \leq M \) for all \( a \leq x \leq b \).

Solution: For us, \( a = -2, b = -1, \) and \( N = 10. \) It remains to find a suitable M. First, note that 
\[ f'(x) = 3x^{-\frac{2}{3}} \quad \text{and} \quad f''(x) = -\frac{2}{3}x^{-\frac{5}{3}}. \]

On the interval \([-2, -1]\), \( f''(x) \) is positive and increasing, therefore, \( |f''(x)| \leq f''(-1) = \frac{2}{3}, \) for \( x \) in \([-2, -1]\). So we take \( M = \frac{2}{3} \) and see that our error is at most 
\[ \frac{\frac{2}{3} \cdot (1)^3}{24 \cdot (10)^2}. \]

Note also that, since \( f''(x) \) is positive on \([-2, -1]\), \( f(x) \) is concave up on this interval, and so \( M_{20} \) is an underestimate for \( \int_{-2}^{-1} 3 \times x^{-\frac{1}{3}} \, dx. \)
Ex. 3: Give an upper bound for the error in approximating
\[ \int_{0}^{\frac{1}{2}} e^{-x^2} \, dx \] by using the Trapezoid Rule with 50 steps.
Use the error formula from Ex. 1.

Sol: For us, \( a = 0, \ b = \frac{1}{2}, \) and \( N = 50. \) It remains to find a suitable \( M. \) First, note that
\[ f'(x) = -2x \cdot e^{-x^2} \quad \text{and} \quad f''(x) = e^{-x^2} (4x^2 - 2). \]
For all \( x \) in \( (0, \frac{1}{2}) \), \( e^{-x^2} \) is positive while \( 4x^2 - 2 \) is negative; thus, \( f''(x) \) is negative on this interval.
We compute \( f''(x) = -4e^{-x^2} \cdot x \cdot (2x^2 - 3) \) and verify that this quantity is positive for all \( x \) in \( (0, \frac{1}{2}) \). Therefore, since the derivative of \( f''(x) \) is positive on this interval, \( f''(x) \) is increasing on \( (0, \frac{1}{2}) \). Since \( f''(x) \) is negative and increasing on \( (0, \frac{1}{2}) \), it attains its minimum value over the interval \([0, \frac{1}{2}]\) at \( x = 0 \).
Thus, \( |f''(x)| \) is maximized at \( x = 0 \).
So we take \( M = |f''(0)| = 2 \), and we see that our error is at most \( \frac{2 \cdot 1^3}{12(50)^2} \).
Let $I = \int_{0}^{1} \cos(x^2) \, dx$. It can be shown that the 4th derivative of $\cos(x^2)$ has absolute value at most 60 on $[0, 1]$. Find $n$ such that the Simpson's rule approximation to $I$ using $n$ points has error at most 0.001. Use the error formula $\left| \int_{a}^{b} f(x) \, dx - S_n \right| \leq \frac{L(b-a)^5}{180 \, n^4}$, where $L$ is a number such that $|f^{(4)}(x)| \leq L$ for all $a \leq x \leq b$.

**Solution:** We want to find a number $n$ such that

$$\frac{L(b-a)^5}{180 \, n^4} = \frac{60}{180 \, n^4} \leq 0.001; \quad \left( L = 60, a = 0, \quad b = 1 \right)$$

This will guarantee that the error from using Simpson's rule is at most 0.001. We simply solve the inequality for $n:\ \frac{60}{180 \, n^4} \leq 0.001 \Rightarrow \frac{180 \, n^4}{60} \geq \frac{1}{0.001} = 1000$.

$$\Rightarrow 3 \, n^4 \geq 1000 \Rightarrow n^4 \geq \frac{1000}{3} \Rightarrow n \geq \left( \frac{1000}{3} \right)^{1/4}.$$ 

So we can use Simpson's rule with any even integer $n$ at least $\left( \frac{1000}{3} \right)^{1/4}$ to guarantee that our approximation is accurate to within 0.001.
Midpoint & Trapezoid Rule! Overestimate or underestimate?

**FACT:** (1) If \( f(x) \) is concave up on \((a, b)\), then
the midpoint rule is an underestimate for \( \int_a^b f(x) \, dx \),
while the trapezoid rule is an overestimate.

(2) If \( f(x) \) is concave down on \((a, b)\), then the
midpoint rule is an overestimate for \( \int_a^b f(x) \, dx \), while
the trapezoid rule is an underestimate.

**Why?** Proof by picture:

**Trapezoid rule:**

\[
\begin{array}{c}
X_{j-1} \\
\hline
X_j
\end{array}
\]

**Midpoint rule:** Only show the concave up statement. Concave down
is similar.

\[
\begin{array}{c}
X_{j-1} \\
\hline
X_j \\
\hline
X_j
\end{array}
\]

rotate top of rectangle until it is tangent to \( f(x) \); this does not change area!