**Power Series**

Definition: A power series in \((x-c)\), or a power series centered at \(c\), is a series of the form
\[
\sum_{n=0}^{\infty} A_n (x-c)^n = A_0 + A_1 (x-c) + A_2 (x-c)^2 + A_3 (x-c)^3 + \ldots
\]

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**Convergence of Power Series**

All power series converge when \(x = c\), the center.
For what other values of \(x\) does \(\sum_{n=0}^{\infty} A_n (x-c)^n\) converge?

Let \(A = \lim_{n \to \infty} \left| \frac{A_{n+1}}{A_n} \right|\).

- If \(A\) is finite and nonzero, set \(R = \frac{1}{A}\).
- If \(A = 0\), set \(R = \infty\). (Interval of convergence is \((-\infty, \infty)\))
- If \(A\) diverges to \(\infty\), set \(R = 0\). (Interval of convergence is \(\{c\}\))

\(R = \text{radius of convergence}\).
\(\sum_{n=0}^{\infty} A_n (x-c)^n\) converges for all \(x\) in \((c-R, c+R)\), and the series may converge at the endpoints \(x = c-R\) or \(x = c+R\); test those values separately. The series diverges for all \(x < c-R\) or \(x > c+R\) (and possibly at the endpoints). The set of \(x\)-values for which the power series converges is called the interval of convergence.
Example: Determine the interval of convergence and the radius of convergence for the power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n \sqrt{n+1}} (x+1)^n.$$ 

**Solution:**

Center: $-4$

$$A = \lim_{n \to \infty} \left| \frac{A_{n+1}}{A_n} \right| = \lim_{n \to \infty} \left| \frac{\frac{(-1)^{n+1}}{3^{n+1} \sqrt{n+2}}}{\frac{(-1)^n}{3^n \sqrt{n+1}}} \right| = \lim_{n \to \infty} \frac{3^n \sqrt{n+1}}{3^{n+1} \sqrt{n+2}}$$

$$= \lim_{n \to \infty} \frac{\sqrt{n+1}}{3 \sqrt{n+2}} = \frac{1}{3} \lim_{n \to \infty} \frac{\sqrt{n+1}}{\sqrt{n+2}} = \frac{1}{3}.$$ 

So the radius of convergence is $R = \frac{1}{A} = 3$.

So this series converges for all $x$ in $(-4-3, -4+3)$, and it might converge at the endpoints, $x = -7, x = -1$.

$x = -7$:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n \sqrt{n+1}} (-3)^n = \sum_{n=0}^{\infty} \left( \frac{-1}{3} \right)^n \frac{1}{\sqrt{n+1}}$$

$$= \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$$

This diverges, by e.g. the limit comparison test with $\sum \frac{1}{n}$.
\[
\begin{align*}
\text{So the interval of convergence is } & \quad (-7, -1]. \\
\text{Alternatively, one can do the ratio test on the entire power series.} \\
\text{Ratio test:} \\
\lim_{n \to \infty} \left| \frac{\frac{(-1)^{n+1}}{3^{n+1} \sqrt{n+1}} \cdot (x+4)^{n+1}}{\frac{(-1)^n}{3^n \sqrt{n+1}} \cdot (x+4)^n} \right| &= \lim_{n \to \infty} \left| \frac{(-1) 3 \sqrt{n+1}}{3^{n+1} \sqrt{n+2}} \right| |x+4| \\
&= \frac{1}{3} |x+4| \lim_{n \to \infty} \frac{\sqrt{n+1}}{\sqrt{n+2}} = \frac{1}{3} |x+4| < 1 \\
\end{align*}
\]

want this for convergence!

\[
\frac{1}{3} |x+4| < 1 \quad \Rightarrow \quad |x+4| < 3 \\
-3 < x+4 < 3 \quad \Rightarrow \quad -7 < x < -1.
\]

Test endpoints separately.
Ex: Consider a power series \( \sum_{n=0}^{\infty} A_n (x-5)^n \).

\( a) \) Suppose it converges at \( x=3 \). Does it converge at \( x=10 \) ?

\( b) \) Suppose it diverges at \( x=15 \). Does it diverge at \( x=-15 \) ?

\( c) \) Does it converge at \( x=14 \)? What can you say about the radius of convergence of the series?

Sol: Plot the interval of convergence:

\[ \left( -3 \right) \quad 5 \quad 10 \]

Yes, it converges at \( x=10 \).

Algebraically:

I know \( \sum_{n=0}^{\infty} A_n (-3-5)^n = \sum_{n=0}^{\infty} A_n (-8)^n \) converges absolutely.

i.e. \( \sum_{n=0}^{\infty} A_n 8^n \) converges.

Does \( \sum_{n=0}^{\infty} A_n (10-5)^n = \sum_{n=0}^{\infty} A_n 5^n \) converge?

Yes, by the comparison test.
b) Plot interval of convergence:

Yes; -15 is further from \( \xi \) the center, than 15 is.

c) Challenge problem!
3.5.2 Working with power series

- Let \( f(x) = \sum_{n=0}^{\infty} A_n (x-c)^n \), \( g(x) = \sum_{n=0}^{\infty} B_n (x-c)^n \)

for all \( x \) such that \( |x-c| < R \) (so the ROC for both \( f \) and \( g \) is at least \( R \)),

- \( f(x) + g(x) = \sum_{n=0}^{\infty} (A_n + B_n) (x-c)^n \)

- \( k \cdot f(x) = \sum_{n=0}^{\infty} kA_n (x-c)^n \)

- \( (x-c)^N \sum_{n=0}^{\infty} A_n (x-c)^n = \sum_{n=0}^{\infty} A_n (x-c)^{n+N} = \sum_{k=N}^{\infty} A_{k-N} (x-c)^k \)

- \( f'(x) = \sum_{n=0}^{\infty} A_n n (x-c)^{n-1} \)

- \( \int_c^x f(t) \, dt = \sum_{n=0}^{\infty} A_n \frac{(x-c)^{n+1}}{n+1} \)

- \( \int f(x) \, dx = C + \sum_{n=0}^{\infty} A_n \frac{(x-c)^{n+1}}{n+1} \), \( C \) arbitrary constant

All of these are true for \( x \) such that \( |x-c| < R \)
All of the right-hand sides have the same radius of convergence as the left-hand sides. (Convergence at endpoints of the interval of convergence may change.)

\[ \text{Recall: For } |x| < 1, \quad \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}. \]

\[ \text{Ex: Find a power series representation for } \frac{1}{1+x^2}, \; |x| < 1. \]

\[ \text{Note: } \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}. \]

\[ \text{Now: For } |x| < 1, \]

\[ \arctan(x) + C = \int \frac{1}{1+x^2} \, dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}. \]