Tricksy Efficient Solutions for $\dot{\vec{x}} = A\vec{x}$

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The Problem Find the general solution for the 2x2 system

$$\dot{\vec{x}} = A\vec{x}, \quad \text{where} \quad A = \begin{bmatrix} a & b \\ c & k \end{bmatrix}.$$

Characteristic Polynomial With A as above, define

$$p(s) = \det\left(A - sI\right) = \begin{vmatrix} a-s & b \\ c & k-s \end{vmatrix} = s^2 - (a+k) + (ak-bc).$$

Key Observations

I. If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is any solution of $\dot{\vec{x}} = A\vec{x}$,

then both scalar components x_1 and x_2 must satisfy

$$\ddot{y} - (a+k)\dot{y} + (ak-bc)y = 0. \qquad (*)$$

(This is the ODE 'reverse-engineered' from $p(s)$ above.)

II. (a) If $b \neq 0$, the first component-equation in $\dot{\vec{x}} = A\vec{x}$ is

$$\dot{x}_1 = ax_1 + bx_2 \iff x_2 = \frac{1}{b}\left(\dot{x}_1 - ax_1\right).$$

So if we know x_1, we can easily find x_2.

(b) If $c \neq 0$, system $\dot{\vec{x}} = A\vec{x}$ includes $\dot{x}_2 = cx_1 + kx_2$,

i.e., $x_1 = \frac{1}{c}\left(\dot{x}_2 - kx_2\right)$. So knowing x_2 makes x_1 easy.

Example 1 : Distinct Real Roots Find the general solution:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Here $A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$ has char. poly. $p(s) = \begin{vmatrix} 2-s & -3 \\ 1 & -2-s \end{vmatrix} = s^2 - 1$, which is also the char. poly. for the scalar ODE

$$\ddot{y} - y = 0. \qquad (*)$$

General sol. for this is $y = c_1 e^{t} + c_2 e^{-t}$, $c_1, c_2 \in \mathbb{R}$.

We can use this for either $x_1(t)$ or $x_2(t)$, then rebuild the other.

Option 1 If $x_1 = c_1 e^{t} + c_2 e^{-t}$, use top eq^n $\dot{x}_1 = 2x_1 - 3x_2$:

$$3x_2 = 2x_1 - \dot{x}_1 = 2(c_1 e^{t} + c_2 e^{-t}) - (c_1 e^{t} - c_2 e^{-t}) = c_1 e^{t} + 3c_2 e^{-t}.$$

So $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c_1 e^{t} + c_2 e^{-t} \\ \frac{1}{3} c_1 e^{t} + c_2 e^{-t} \end{bmatrix} = c_1 e^{t} \begin{bmatrix} 1 \\ 1/3 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $c_1, c_2 \in \mathbb{R}$.

Option 2 If $x_2 = c_1 e^{t} + c_2 e^{-t}$, use lower eq^n $\dot{x}_2 = x_1 - 2x_2$:

$$x_1 = +2x_2 + \dot{x}_2 = +2(c_1 e^{t} + c_2 e^{-t}) + (c_1 e^{t} - c_2 e^{-t}) = +3c_1 e^{t} + c_2 e^{-t}.$$

So $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $c_1, c_2 \in \mathbb{R}$.

Discussion ① Options 1 and 2 are compatible, but they impose different interpretations on the arbitrary constants c_1, c_2.

② Eigenvalues $+1, -1$ with eigenvectors $\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ emerge organically!

Example 2: Root with Nonzero Imaginary Parts Solve:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Here $A = \begin{bmatrix} -1 & 2 \\ -1 & -3 \end{bmatrix}$ has char. poly. $p(s) = \begin{vmatrix} -1-s & 2 \\ -1 & -3-s \end{vmatrix} = s^2+4s+5$,

which is also associated with the scalar ODE

$$\ddot{y} + 4\dot{y} + 5y = 0.$$

Roots $0 = p(s) = (s+2)^2 + 1 \iff s = -2 \pm i$ reveal one complex-valued solution $y = e^{(-2+i)t}$. Use this for either x_1 or x_2, then back-substitute for the other.

Option 1. If $x_2(t) = e^{(-2+i)t}$, then system says $\dot{x}_2 = -x_1 - 3x_2$, so

$$x_1 = -3x_2 - \dot{x}_2 = -3e^{(-2+i)t} - (-2+i)e^{(-2+i)t} = (-1-i)e^{(-2+i)t}.$$

So a vector solution for $\dot{\vec{x}} = A\vec{x}$ (with complex values) is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = e^{(-2+i)t}\begin{bmatrix} -1-i \\ 1 \end{bmatrix} = e^{-2t}(\cos t + i\sin t)\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} + i\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right)$$

$$= e^{-2t}\left(\begin{bmatrix} \sin t - \cos t \\ \cos t \end{bmatrix} + i\begin{bmatrix} -\sin t - \cos t \\ \sin t \end{bmatrix}\right).$$

Both the real part and the imaginary part, separately, are solutions for the original system. So use them to construct the general sol:

$$\vec{x}(t) = c_1 e^{-2t}\begin{bmatrix} \sin t - \cos t \\ \cos t \end{bmatrix} + c_2 e^{-2t}\begin{bmatrix} -\sin t - \cos t \\ \sin t \end{bmatrix}, \quad c_1, c_2 \in \mathbb{R}.$$

<u>Option 2</u> Using $x_1(t) = e^{(-2+i)t}$ instead leads, via $\dot{x}_1 = -x_1 + 2x_2$, to

$$x_2(t) = \frac{1}{2}\left[x_1 + \dot{x}_1\right] = \frac{1}{2}\left[e^{(-2+i)t} + (-2+i)e^{(-2+i)t}\right] = \frac{1}{2}(-1+i)e^{(-2+i)t}.$$

So a vector-valued complex solution is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e^{(-2+i)t} \\ \frac{1}{2}(-1+i)e^{(-2+i)t} \end{bmatrix}.$$

Another, more convenient, one is 2 times this ... let's switch to

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = e^{(-2+i)t}\begin{bmatrix} 2 \\ -1+i \end{bmatrix} = e^{-2t}(\cos t + i\sin t)\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix} + i\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$= e^{-2t}\begin{bmatrix} 2\cos t \\ -\cos t - \sin t \end{bmatrix} + i\, e^{-2t}\begin{bmatrix} 2\sin t \\ \cos t - \sin t \end{bmatrix}.$$

Both real part and imag part, separately, solve our system. Combine these independent parts to get the general solution

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = k_1 e^{-2t}\begin{bmatrix} 2\cos t \\ -\cos t - \sin t \end{bmatrix} + k_2 e^{-2t}\begin{bmatrix} 2\sin t \\ \cos t - \sin t \end{bmatrix}, \quad k_1, k_2 \in \mathbb{R}.$$

<u>Reconciliation</u> Forms of general solution above are rather different in appearance, but in fact they describe <u>identical lists of functions</u>. To see this, plug $c_1 = k_2 - k_1$ and $c_2 = -k_1 - k_2$ into the first setup to recover the second; or, use $k_1 = -\left(\frac{c_1+c_2}{2}\right)$ with $k_2 = \frac{c_1 - c_2}{2}$ to transform the second into the first.

Example 3: Repeated Real Roots

Solve:

$$\dot{\vec{x}} = A\vec{x}, \qquad A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}.$$

Here $p(s) = \det(A - sI) = \quad ((1-s)(3-s) - (-1) = s^2 - 4s + 4 = (s-2)^2,$

so $x_1(t)$ must have form $\quad x_1(t) = c_1 e^{2t} + c_2 t e^{2t}$ for some c_1, c_2. Then any compatible $x_2(t)$ must obey the first ODE,

$$\dot{x}_1 = x_1 - x_2, \quad \text{i.e.,} \quad x_2 = x_1 - \dot{x}_1:$$

$$x_2 = \left(c_1 e^{2t} + c_2 t e^{2t}\right) - \left(2c_1 e^{2t} + c_2 e^{2t} + 2c_2 t e^{2t}\right)$$

$$= c_1 e^{2t}(-1) + c_2 e^{2t}(-1 - t).$$

Packing components into a vector gives the general solution:

$$\vec{x}(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} t \\ -1-t \end{bmatrix}, \qquad c_1, c_2 \in \mathbb{R}.$$

Notes: (1) The same family of functions can be displayed in several different-<u>looking</u> ways. Substitute $c_1 = k_1 - r k_2$ and $c_2 = k_2$ with any constant r to get the alternative style

$$\vec{x}(t) = k_1 e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + k_2 e^{2t} \begin{bmatrix} t - r \\ -t - 1 + r \end{bmatrix}, \qquad k_1, k_2 \in \mathbb{R}.$$

(2) Eigenvalue $s = 2$ with eigenvector $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is visible in answer.

MATRIX EXPONENTIAL EXTENSIONS

It's straightforward to construct e^{At} once we have the general solution of $\dot{\vec{x}} = A\vec{x}$. Each column of e^{At} is just one of the functions $\vec{x}(t)$ listed in the general solution, chosen so that the initial condition $e^{At}\big|_{t=0} = I$ works. Here are the three matrices above for an encore.

<u>Eg 1</u> Find e^{At} for $A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$.

Each column has form $\vec{x} = \begin{bmatrix} c_1 e^{t} + c_2 e^{-t} \\ \frac{1}{3}c_1 e^{t} + c_2 e^{-t} \end{bmatrix}$ for some c_1, c_2.

Col #1 needs $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{x}(0) = \begin{bmatrix} c_1 + c_2 \\ \frac{1}{3}c_1 + c_2 \end{bmatrix}$, i.e., $c_1 = 3/2$, $c_2 = -1/2$.

So col #1 is $\begin{bmatrix} 3/2\, e^{t} - 1/2\, e^{-t} \\ 1/2\, e^{t} - 1/2\, e^{-t} \end{bmatrix}$.

Col #2 needs different c_1, c_2, to make $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \vec{x}(0) = \begin{bmatrix} c_1 + c_2 \\ \frac{1}{3}c_1 + c_2 \end{bmatrix}$

So $c_1 = \frac{-3}{2}$ but $c_2 = +3/2$, and col #2 is $\begin{bmatrix} -3/2\, e^{t} + 3/2\, e^{-t} \\ -1/2\, e^{t} + 3/2\, e^{-t} \end{bmatrix}$.

Stacking these columns in position completes the work:

$$e^{At} = \begin{bmatrix} \frac{1}{2}\left(3e^{t} - e^{-t}\right) & -\frac{3}{2}\left(e^{t} - e^{-t}\right) \\ \frac{1}{2}\left(e^{t} - e^{-t}\right) & -\frac{1}{2}\left(e^{t} - 3e^{-t}\right) \end{bmatrix}.$$

Eg 2 Given $A = \begin{bmatrix} -1 & 2 \\ -1 & -3 \end{bmatrix}$, find e^{At}.

Gen sol $\vec{x}(t) = c_1 e^{-2t} \begin{bmatrix} \sin t - \cos t \\ \cos t \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} -\sin t - \cos t \\ \sin t \end{bmatrix}$

sets the pattern for the cols.

Col #1: $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{x}(0) = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -c_1 - c_2 \\ c_1 \end{bmatrix}$

requires $c_1 = 0$, $c_2 = -1$. Simplify:

col #1: $e^{-2t} \begin{bmatrix} \sin t + \cos t \\ -\sin t \end{bmatrix}$.

Col #2: $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \vec{x}(0) = \begin{bmatrix} -c_1 - c_2 \\ c_1 \end{bmatrix}$ requires $c_1 = 1$, $c_2 = -1$. So

col #2: $e^{-2t} \begin{bmatrix} (\sin t - \cancel{\cos t}) - (-\sin t - \cancel{\cos t}) \\ (\cos t) - (\sin t) \end{bmatrix}$

Assemble cols above and report:

$$e^{At} = \begin{bmatrix} e^{-2t}(\sin t + \cos t) & 2e^{-2t} \sin t \\ -e^{-2t} \sin t & e^{-2t}(\cos t - \sin t) \end{bmatrix}.$$

Eg 3 If $A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$, find $\mathcal{L}^{-1}\{(sI-A)^{-1}\}$.

Since $\mathcal{L}\{e^{At}\} = (sI-A)^{-1}$, this is just a cryptic way of requesting e^{At}. The cols of e^{At} are known to have form found above:

$$\vec{x}(t) = c_1 e^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} t \\ -1-t \end{bmatrix}, \qquad c_1, c_2 \in \mathbb{R}.$$

For e^{At} col #1, $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{x}(0) = \begin{bmatrix} c_1 \\ -c_1 - c_2 \end{bmatrix} \Leftrightarrow c_1 = 1,\ c_2 = -1$

so col #1 is $\begin{bmatrix} e^{2t}(1-t) \\ e^{2t}(-1+1+t) \end{bmatrix} = \begin{bmatrix} e^{2t}(1-t) \\ te^{2t} \end{bmatrix}$

For e^{At} col #2, $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \vec{x}(0) = \begin{bmatrix} c_1 \\ -c_1 - c_2 \end{bmatrix} \Leftrightarrow c_1 = 0,\ c_2 = -1,$

so col #2 is $\begin{bmatrix} -te^{2t} \\ (1+t)e^{2t} \end{bmatrix}$.

Combine cols to report

$$e^{At} = \begin{bmatrix} (1-t)e^{2t} & -te^{2t} \\ te^{2t} & (1+t)e^{2t} \end{bmatrix}.$$

LT practice: $sI - A = \begin{bmatrix} s-1 & 1 \\ -1 & s-3 \end{bmatrix} \Rightarrow (sI-A)^{-1} = \dfrac{1}{(s-2)^2}\begin{bmatrix} s-3 & -1 \\ 1 & s-1 \end{bmatrix}$.

In top left corner, $\mathcal{L}^{-1}\left\{\frac{s-3}{(s-2)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2} - \frac{1}{(s-2)^2}\right\} = e^{2t}\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s^2}\right\} = e^{2t}(1-t)$.

(YAY.)

NONHOMOGENEOUS SYSTEMS

For the system $\dot{\vec{x}} = A\vec{x} + \vec{g}(t)$, we have a formula:

$$\circledast \qquad \vec{x}(t) = e^{At}\vec{x}(0) + \int_0^t e^{A(t-r)}\vec{g}(r)\,dr.$$

Actually using this is conceptually straightforward, but algebraically onerous.

Eg 1 Detail $\circledast$ for $A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$.

We know $$e^{At} = \frac{1}{2}\begin{bmatrix} 3e^{t}-e^{-t} & e^{-t}-e^{t} \\ e^{t}-e^{-t} & 3e^{-t}-e^{t} \end{bmatrix}$$

so $$e^{-Ar} = \frac{1}{2}\begin{bmatrix} 3e^{-r}-e^{r} & e^{r}-e^{-r} \\ e^{-r}-e^{r} & 3e^{r}-e^{-r} \end{bmatrix}$$

and $$\vec{x}(t) = \frac{1}{2}\begin{bmatrix} 3e^{t}-e^{-t} & e^{-t}-e^{t} \\ e^{t}-e^{-t} & 3e^{-t}-e^{t} \end{bmatrix}\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

$$+ \frac{1}{2}\begin{bmatrix} 3e^{t}-e^{-t} & e^{-t}-e^{t} \\ e^{t}-e^{-t} & 3e^{-t}-e^{t} \end{bmatrix}\int_0^t \frac{1}{2}\begin{bmatrix} 3e^{-r}-e^{r} & e^{r}-e^{-r} \\ e^{-r}-e^{r} & 3e^{r}-e^{-r} \end{bmatrix}\begin{bmatrix} g_1(r) \\ g_2(r) \end{bmatrix}dr$$

$$= \begin{bmatrix} (3e^{t}-e^{-t})\frac{x_1(0)}{2} + (e^{-t}-e^{t})\frac{x_2(0)}{2} \\ (e^{t}-e^{-t})\frac{x_1(0)}{2} + (3e^{-t}-e^{t})\frac{x_2(0)}{2} \end{bmatrix}$$

column vector containing fcns

$$+ \frac{1}{2}\begin{bmatrix} 3e^{t}-e^{-t} & e^{-t}-e^{t} \\ e^{t}-e^{-t} & 3e^{-t}-e^{t} \end{bmatrix}\int_0^t \begin{bmatrix} (3e^{-r}-e^{r})\frac{g_1(r)}{2} + (e^{r}-e^{-r})\frac{g_2(r)}{2} \\ (e^{-r}-e^{r})\frac{g_1(r)}{2} + (3e^{r}-e^{-r})\frac{g_2(r)}{2} \end{bmatrix}dr$$

2x2 mtx

integral gives vector, 2x1

Methods above, where we reduce a given 2×2 problem to a single second-order ODE … now nonhomogeneous … will work in this case also. It can be a little easier than the vector-matrix approach, but it's still not 'easy'. The textbook of Boyce/diPrima shows 4 approaches on the same problem:

(i) eigenvalue / eigenvector inspired coordinate change
(ii) method of undetermined coeffs
(iii) variation of parameters
(iv) Laplace transform.

They skip the general formula ⊗ quoted above. Here is a solution by the scalar method introduced above.

Example (Boyce/DiPrima, Elementary DE's & BVP's, 10th ed, §7.9). Find gen. sol:

$$\dot{\vec{x}} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \vec{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix}$$

Sol Expand

$$(1) \quad \dot{x}_1 = -2x_1 + x_2 + 2e^{-t}$$
$$(2) \quad \dot{x}_2 = x_1 - 2x_2 + 3t$$

Differentiate through (1), sub from (2):

$$\ddot{x}_1 = -2\dot{x}_1 + \dot{x}_2 - 2e^{-t}$$
$$= -2\dot{x}_1 + (x_1 - 2x_2 + 3t) - 2e^{-2t}$$

Go back to (1) to isolate x_2,

$$(1') \quad x_2 = \dot{x}_1 + 2x_1 - 2e^{-t}$$

and use this in the calculation in progress above:

$$\ddot{x}_1 = -2\dot{x}_1 + x_1 - 2\left[\dot{x}_1 + 2x_1 - 2e^{-t}\right] + 3t - 2e^{-2t}$$

$$(\star\star) \quad \Leftrightarrow \quad \underbrace{\ddot{x}_1 + 4\dot{x}_1 + 3x_1} = 2e^{-t} + 3t$$

Define operator $L[x_1]$. Note $p(s) = s^2 + 4s + 3 = (s+1)(s+3)$, so complementary eqn $L[x_1] = 0$ has gen sol $x_1^{(c)} = c_1 e^{-3t} + c_2 e^{-t}$.

To solve $L[x_1] = 3t$, guess $x_1 = Mt + N$. Plug in:

$$0 + 4M + 3(Mt + N) = 3t \quad \Leftrightarrow \quad M = 1, \ N = -4/3.$$

So another solution element is $x_1^{(p1)} = t - \frac{4}{3}$.

To solve $L[x_1] = 2e^{-t}$, use exp. shift: $x_1 = e^{-t}u(t)$ will do this iff

$$e^{-t}\left(\ddot{u} + 2\dot{u} + 0u\right) = 2e^{-t}, \text{ i.e., } \quad \ddot{u} + 2\dot{u} = 2.$$

Function $u(t) = +t$ does this, so we have $x_1^{(p2)}(t) = +te^{-t}$.

So the general solution of (✱✱) is

$$x_1(t) = x_1^{(c)} + x_1^{(p1)} + x_1^{(p2)} = c_1 e^{-3t} + c_2 e^{-t} + t + te^{-t} - 4/3.$$

The companion component comes from line (1′) above:

$$\begin{aligned}
x_2 &= 2x_1 + \dot{x}_1 - 2e^{-t} \\
&= 2c_1 e^{-3t} + 2c_2 e^{-t} + 2t + 2te^{-t} - 8/3 && \leftarrow 2x_1 \\
&\quad -3c_1 e^{-3t} - c_2 e^{-t} \qquad \xrightarrow{\text{(prod rule)}} -te^{-t} + 1 && \leftarrow \dot{x}_1 \\
&\quad + e^{-t} \\
&\quad -2e^{-t} && \leftarrow -2e^{-t} \\
&= c_1\left(-e^{-3t}\right) + c_2\left(e^{-t}\right) + 2t + te^{-t} - e^{-t} - 5/3
\end{aligned}$$

OOPS: no c_2 factor — should get separate column

The final answer in vector form:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{-t}\begin{bmatrix} 0 \\ -1 \end{bmatrix} + te^{-t}\begin{bmatrix} +1 \\ +1 \end{bmatrix} + \begin{bmatrix} t - 4/3 \\ 2t - 5/3 \end{bmatrix}.$$

<u>Suggestion:</u> Try finding gen sol for x_1 in (✱✱) using Laplace Transform instead of method used here.

<u>Philosophy Question</u> If we can

(i) solve $\dot{\vec{x}} = A\vec{x}$

(ii) find e^{At}

without any reference to eigenvalues/eigenvectors, why mention those at all, ever?

<u>Answers</u> ① For matrices A of size 3x3 and bigger, the 'easy' methods shown here get much more elaborate. The approach based on eigen-analysis allows a systematic + efficient treatment for larger problems.

② Remember the phase plane pictures. The eigenvalues and eigenvectors carry important insights into the system behaviour and stability, even in the 2x2 case.

<u>Learning/Teaching Strategy</u> Is it reasonable to stress eigen-analysis in 2D, since the direct alternative is quicker?

<u>Answers</u> ① Probably not, except for the insights mentioned in ② above.

② We should probably put more time and effort into 3x3 and larger problems, where the eigenstuff really pays off. Next year!