II. Fourier Series

$\boxed{\text{7 Jan '05}}$

A. Signals; Periodicity

A *signal* is a complex-valued function of one real variable—typically, $f = f(t) = u(t) + iv(t)$ for $u, v : \mathbb{R} \to \mathbb{R}$.

Stereo. Think of $u(t) = \mathrm{Re}\,\{f(t)\}$ and $v(t) = \mathrm{Im}\,\{f(t)\}$ as independent channels of a single signal, like stereo.

Periodicity. A signal f is *periodic* if some $T > 0$ obeys

$$(*) \qquad f(t+T) = f(t) \text{ for all } t.$$

Say "f is T-periodic" when this happens. Sketch some examples.

Minimality. If f is T-periodic, it is automatically $2T$-periodic:

$$f(t+2T) = f([t+T]+T) = f(t+T) = f(t).$$

Indeed, f is nT-periodic for every $n = 1, 2, 3, \ldots$. The smallest $T > 0$ compatible with $(*)$ is the *fundamental period* for f; it's typically a time in seconds.

B. Harmonic Oscillators

A simple periodic function (sketch) is

$$u(t) = A\cos(\omega t - \phi).$$

Amplitude $A \geq 0$, angular frequency $\omega > 0$, phase $\phi \in \mathbb{R}$. Fcn cos is periodic with fundamental period 2π, so fundamental period T for u obeys $u(t+T) = u(t)$ with

$$\omega(t+T) - \phi = [\omega t - \phi] + 2\pi, \quad \text{i.e.,} \quad T = \frac{2\pi}{\omega}.$$

Here $\omega = 2\pi/T$ is the **angular frequency** in **rad/sec**. Engineers measure cycle-frequency in **Hz = cycle/sec**: this is $f = 1/T = \omega/(2\pi)$. Typical values: AC hum is 60 Hz; Cello's open C-string is 65.4 Hz; Violin A is 440 Hz; Squeak is 10^4 Hz; AM 600 is 6×10^5 Hz; CFOX is 9.73×10^7 Hz; My Computer is 2.8×10^9 Hz; Light oscillates at ??? Hz.

Alternate form. $u(t) = a\cos(\omega t) + b\sin(\omega t)$.

Reconciliation. Trig identity $\Rightarrow$

$$A\cos(\omega t - \phi) = A\cos(\phi)\cos(\omega t) + A\sin(\phi)\sin(\omega t).$$

Matching:

$$a = A\cos(\phi),\ b = A\sin(\phi), \qquad \text{i.e.,} \qquad A = \sqrt{a^2+b^2},\ (\cos\phi, \sin\phi) = \left(\frac{a}{A}, \frac{b}{A}\right).$$

Complex Representations.

$$\left.\begin{array}{lll} e^{i\theta} & & = \cos(\theta) + i\sin(\theta) \\ e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) & & = \cos(\theta) - i\sin(\theta) \end{array}\right] \implies \left[\begin{array}{l} \cos(\theta) = \dfrac{e^{i\theta}+e^{-i\theta}}{2}, \\ \sin(\theta) = \dfrac{e^{i\theta}-e^{-i\theta}}{2i}. \end{array}\right.$$

Two Channel Oscillator with Shared Frequency. Choose same frequency ω in two channels, writing

$$u(t) = a_1\cos(\omega t) + b_1\sin(\omega t), \qquad v(t) = a_2\cos(\omega t) + b_2\sin(\omega t).$$

Note that a_1, b_1, a_2, b_2 can be arbitrary real numbers; let

$$f(t) = u(t) + iv(t). \tag{1}$$

In complex-exponential form,

$$\begin{aligned} f(t) &= \frac{a_1+ia_2}{2}\left(e^{i\omega t}+e^{-i\omega t}\right) + \frac{b_1+ib_2}{2i}\left(e^{i\omega t}-e^{-i\omega t}\right) \\ &= \left(\frac{(a_1+b_2)+i(a_2-b_1)}{2}\right)e^{i\omega t} + \left(\frac{(a_1-b_2)+i(a_2+b_1)}{2}\right)e^{-i\omega t} \\ &\stackrel{\text{def}}{=} c_1e^{i\omega t} + c_{-1}e^{-i\omega t}. \end{aligned} \tag{2}$$

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Here $c_1 = \dfrac{(a_1+b_2)+i(a_2-b_1)}{2}$, $c_{-1} = \dfrac{(a_1-b_2)+i(a_2+b_1)}{2}$. The trigonometric expression in (1) and the exponential expression in (2) carry the same information (each involves four real constants), but (2) is far more convenient.

Important Notes:

1. Signal f is real-valued ($v \equiv 0$ on complex channel)
$\Leftrightarrow a_2 = 0 = b_2 \Leftrightarrow c_1 = (a_1 - ib_1)/2,\ c_{-1} = (a_1+ib_1)/2 \Leftrightarrow c_{-1} = \overline{c_1}$.

2. Signal f is even (pure cosines only in both channels)
$\Leftrightarrow b_1 = 0 = b_2 \Leftrightarrow c_1 = \frac{1}{2}(a_1+ia_2),\ c_{-1} = \frac{1}{2}(a_1+ia_2) \Leftrightarrow c_{-1} = c_1$.

3. Signal f is odd (pure sines only in both channels)
$\Leftrightarrow a_1 = 0 = a_2 \Leftrightarrow c_1 = \frac{1}{2}(b_2 - ib_1),\ c_{-1} = \frac{1}{2}(-b_2+ib_1) \Leftrightarrow c_{-1} = -c_1$.

C. Fourier Synthesizer [Signal Generator]

Pick any $\omega > 0$ and use it throughout this discussion. Imagine a device that generates different periodic signals with fundamental angular frequency ω. The signal is determined by a doubly-infinite sequence of complex numbers:

Input: any complex sequence
$(c_k)_{k\in\mathbb{Z}} = (\ldots, c_{-2}, c_{-1}, c_0, c_1, c_2, \ldots)$

$$\downarrow$$

FOURIER
SYNTHESIZER

$$\downarrow$$

Output: the signal
$$\widetilde{f}(t) = c_0 + \left(c_1 e^{i\omega t} + c_{-1}e^{-i\omega t}\right) + \left(c_2 e^{i2\omega t} + c_{-2}e^{-i2\omega t}\right) + \cdots = \sum_{k=-\infty}^{\infty} c_k e^{ik\omega t}.$$

The input values (entries in sequence $\mathbf{c}$) are called the *Fourier coefficients* in $\widetilde{f}$.

Technicalities. 1. For each $t \in \mathbb{R}$, the value $\widetilde{f}(t)$ is defined by this limit:

$$\widetilde{f}(t) = \lim_{N\to\infty} \sum_{k=-N}^{N} c_k e^{ik\omega t}.$$

2. We allow only input "vectors" $\mathbf{c}$ that make this number **finite**:

$$\sum_{k=-\infty}^{\infty} |c_k|^2 = \lim_{N\to\infty} \sum_{k=-N}^{N} |c_k|^2.$$

(This is automatic if $c_k = 0$ for all $k \in \mathbb{Z}$ with a finite number of exceptions.)

Online Demonstration (strongly recommended). Try the applet at this address. It makes sounds as well as pictures:

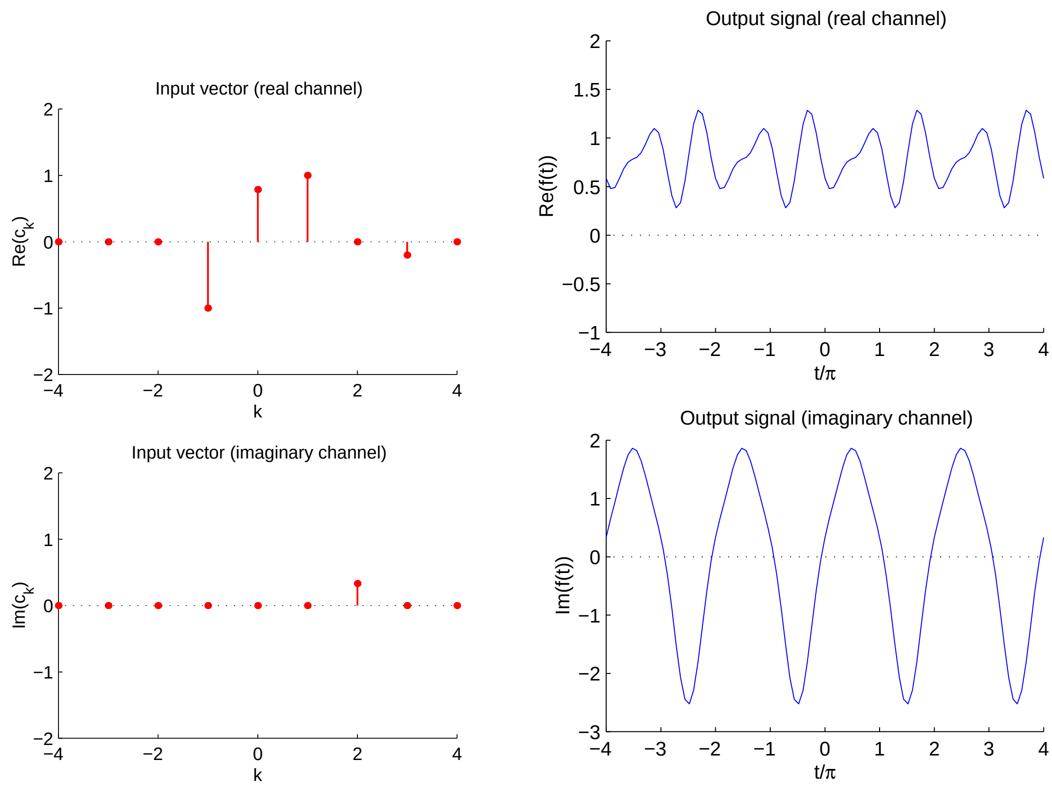
`http://www.phy.ntnu.edu.tw/java/sound/sound.html` .

Calculated Example. Let $\omega = 1$. Find and plot the synthesizer output if the input vector $\mathbf{c} = (c_k)_{k=-\infty}^{\infty}$ has all $c_k = 0$ except for these:

$$c_{-1} = -1,\ c_0 = \pi/4,\ c_1 = 1,\ c_2 = \frac{i}{3},\ c_3 = -\frac{1}{5}.$$

Solution. By Euler's formula, the given choices produce the 2π-periodic signal

$$\begin{aligned}
\widetilde{f}(t) &= \sum_{k=-\infty}^{\infty} c_k e^{ikt} \\
&= \cdots + 0 + c_{-1}e^{-it} + c_0 + c_1 e^{it} + c_2 e^{2it} + c_3 e^{3it} + 0 + \cdots \\
&= -e^{-it} + \frac{\pi}{4} + e^{it} + \frac{i}{3}e^{2it} - \frac{1}{5}e^{3it} \\
&= \left[\frac{\pi}{4} - \frac{1}{3}\sin(2t) - \frac{1}{5}\cos(3t)\right] + i\left[2\sin(t) + \frac{1}{3}\cos(2t) - \frac{1}{5}\sin(3t)\right]
\end{aligned}$$



Properties of $\widetilde{f}$. Our single-frequency study above makes these believable:

1. Signal $\widetilde{f}$ is real-valued (i.e., $\mathrm{Im}(f(t)) = 0$ always) $\Leftrightarrow$ $c_{-k} = \overline{c_k}$ for each $k \in \mathbb{Z}$.

2. Signal $\widetilde{f}$ is even (i.e., $f(-t) = f(t)$ always) $\Leftrightarrow$ $c_{-k} = c_k$ for each $k \in \mathbb{Z}$.

3. Signal $\widetilde{f}$ is odd (i.e., $f(-t) = -f(t)$ always) $\Leftrightarrow$ $c_{-k} = -c_k$ for each $k \in \mathbb{Z}$.

4. f is T-periodic, with $T = 2\pi/\omega$.
[Reason: each summand is T-periodic … $e^{in\omega[t+T]} = e^{in\omega t}e^{in\omega(2\pi/\omega)} = e^{in\omega t}$.]

D. Fourier Analyzer [Spectrometer]

Code-breaking game: a periodic signal with fundamental frequency ω is given. We suspect it is being generated by a Fourier Synthesizer, i.e., it has the form

$$\widetilde{f}(t) = \sum_{k=-\infty}^{\infty} c_k e^{ik\omega t}$$

for some vector of Fourier coefficients $\mathbf{c} = (\ldots, c_{-1}, c_0, c_1, \ldots)$.

Question. What is $\mathbf{c}$?

Answer. Extract one component at a time. Focus on subscript m. To find c_m, notice that for any $k \in \mathbb{Z}$,

$$k \neq m \implies \qquad \int_0^T e^{ik\omega t} e^{-im\omega t}\, dt = \int_0^T e^{i(k-m)\omega t}\, dt = \left.\frac{e^{i(k-m)\omega t}}{i(k-m)\omega}\right|_{t=0}^{T} = 0,$$

(because $\omega T = 2\pi$ makes $e^{i(k-m)\omega T} = 1 = e^0$)

$$k = m \implies \qquad \int_0^T e^{ik\omega t} e^{-im\omega t}\, dt = \int_0^T 1\, dt = T.$$

This is an "orthogonality property" of the family of functions

$$\left\{\ldots, e^{-i2\omega t}, e^{-i\omega t}, 1, e^{i\omega t}, e^{i2\omega t}, \ldots\right\}.$$

Exploit it: multiply $\widetilde{f}(t)$ shown above by $e^{-im\omega t}$ and integrate:

$$\begin{aligned}
\int_0^T \widetilde{f}(t) e^{-im\omega t}\, dt &= \int_0^T \left(\sum_{k=-\infty}^{\infty} c_k e^{ik\omega t}\right) e^{-im\omega t}\, dt \\
&= \sum_{k=-\infty}^{\infty} c_k \left(\int_0^T e^{ik\omega t} e^{-im\omega t}\, dt\right) \\
&= \cdots + 0 + 0 + c_m T + 0 + 0 + \cdots.
\end{aligned}$$

This gives a formula valid for each $m \in \mathbb{Z}$:

$$c_m = \frac{1}{T}\int_0^T \widetilde{f}(t) e^{-im\omega t}\, dt.$$

Now remember that $\omega = 2\pi/T$. When we define $\ell = T/2$, so the period is $T = 2\ell$, we have $\omega = \pi/\ell$ and

$$\boxed{\widetilde{f}(t) = \sum_{k=-\infty}^{\infty} c_k e^{ik\pi t/\ell} \qquad \iff \qquad c_m = \frac{1}{2\ell}\int_0^{2\ell} \widetilde{f}(t) e^{-im\pi t/\ell}\, dt \quad \text{for each } m \in \mathbb{Z}.}$$

Spectral Analysis. Imagine the coefficient-extraction process as a machine. The period $T > 0$ is fixed in advance.

$$\boxed{\begin{array}{l}\text{Input: a } T\text{-periodic signal} \\ \qquad\qquad\qquad f = f(t)\end{array}}$$

$$\downarrow$$

$$\boxed{\text{THE FOURIER ANALYZER}}$$

$$\downarrow$$

$$\boxed{\begin{array}{l}\text{Output: The sequence } \mathbf{c} = (c_k)_{k=-\infty}^{\infty}\text{, where} \\ \qquad c_m = \dfrac{1}{T}\displaystyle\int_0^T f(t)e^{-im\omega t}\,dt,\ m \in \mathbb{Z}.\end{array}}$$

Summary. The Fourier Analyzer reverses the operation of the Fourier Synthesizer. If we put a sequence $\mathbf{c}$ of Fourier coefficients into the synthesizer, a signal $\widetilde{f}$ comes out. Putting that signal $\widetilde{f}$ into the analyzer will reproduce the original coefficients:

$$\boxed{\begin{array}{c}\text{sequence}\\ \mathbf{c} = (c_k)\end{array}} \to \boxed{\begin{array}{l}\text{FOURIER}\\ \text{SYNTHESIZER}\end{array}} \to \boxed{\begin{array}{l}\text{signal}\\ \widetilde{f}\end{array}} \to \boxed{\begin{array}{l}\text{FOURIER}\\ \text{ANALYZER}\end{array}} \to \boxed{\begin{array}{c}\text{sequence}\\ \mathbf{c} = (c_k)\end{array}}.$$

E. Pointwise Convergence

We continue to work with a specific angular frequency $\omega > 0$, using the definitions $T = 2\pi/\omega$ (the fundamental period) and $\ell = T/2$ (the half-period).

We can put any T-periodic signal f into the Fourier analyzer, whether it is generated by the synthesizer or not. The analyzer will produce a vector of Fourier coefficients. These obviously contain some information about f, but how much? Schematically, what will we get when we put f through the following process?

$$\boxed{\begin{array}{l}\text{Signal}\\ f(t)\end{array}} \to \boxed{\begin{array}{l}\text{FOURIER}\\ \text{ANALYZER}\end{array}} \to \boxed{\begin{array}{c}\text{sequence}\\ \mathbf{c} = (c_k)\end{array}} \to \boxed{\begin{array}{l}\text{FOURIER}\\ \text{SYNTHESIZER}\end{array}} \to \boxed{\begin{array}{l}\text{Signal}\\ \widetilde{f}(t) = \displaystyle\sum_{k=-\infty}^{\infty} c_k e^{ik\omega t}\end{array}}.$$

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Example. Suppose $T = 2\pi$. Let f be 2π-periodic and obey $f(t) = t$ for $0 < t < 2\pi$. Find the Fourier coefficients for f and plot the Fourier reconstruction $\widetilde{f}$ corresponding to these coefficients on the interval $[-4\pi, 4\pi]$.

Solution. It's nice to have a sketch of f. The reader is asked to supply this. Notice that the numerical values of $f(0), f(\pm 2\pi), f(\pm 4\pi), \ldots,$ are not specified in the problem statement. They are not needed, because isolated point-values of f have no influence on the values of integrals involving f.

The Fourier coefficients $\mathbf{c} = (c_m)_{m \in \mathbb{Z}}$ are given by

$$c_m = \frac{1}{2\pi} \int_0^{2\pi} t e^{-imt}\, dt, \qquad m \in \mathbb{Z}.$$

Integration by parts gives this general formula, valid whenever $a \neq 0$:

$$\int t e^{at}\, dt = t\left(\frac{e^{at}}{a}\right) - \int \left(\frac{e^{at}}{a}\right) dt = \frac{t}{a} e^{at} - \frac{1}{a^2} e^{at},$$

Using this with $a = -im$ (provided $m \neq 0$) gives

$$c_m = \frac{1}{2\pi}\left[\left(\frac{t}{-im} - \frac{1}{i^2 m^2}\right) e^{-imt}\right]_{t=0}^{2\pi} = \frac{1}{2\pi}\left[\left(-\frac{2\pi}{im} + \frac{1}{m^2}\right) - \left(\frac{1}{m^2}\right)\right] = \frac{i}{m}.$$

A separate calculation is needed for the case where $m = 0$: since $e^{i0t} = 1$,

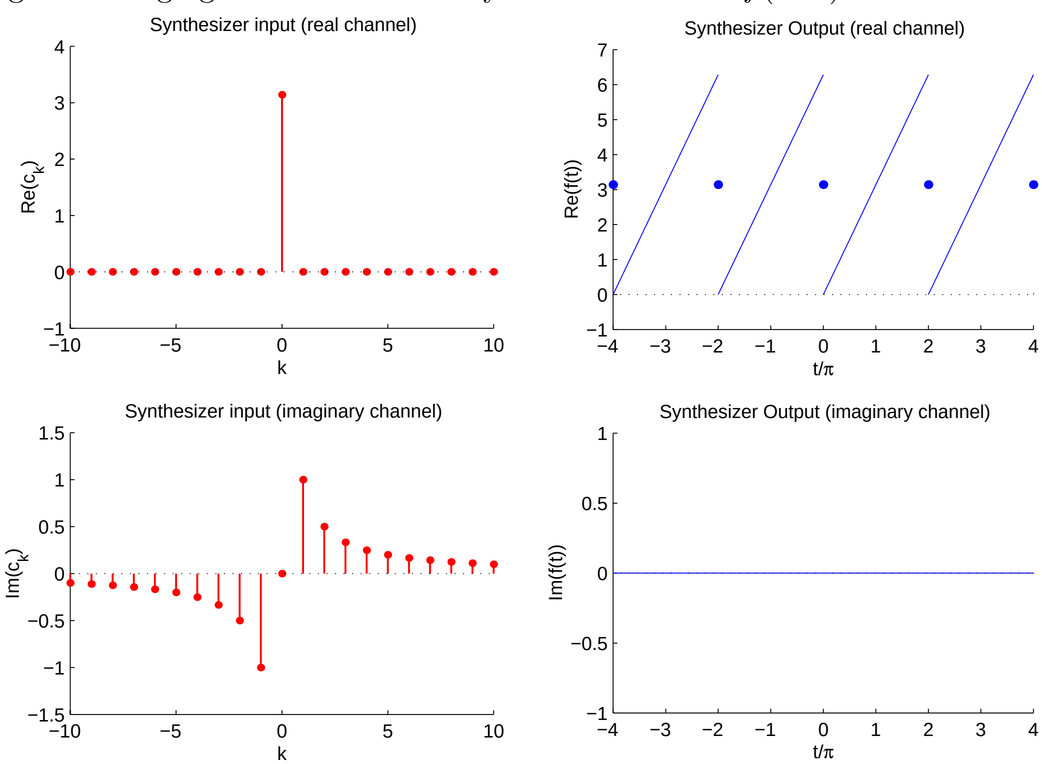
$$c_0 = \frac{1}{2\pi} \int_0^{2\pi} t\, dt = \pi.$$

Combining these results tells us the vector of Fourier coefficients:

$$\mathbf{c} = (c_m)_{m \in \mathbb{Z}}, \qquad \text{where} \qquad c_m = \begin{cases} \pi, & \text{if } m = 0, \\ i/m, & \text{if } m \neq 0. \end{cases}$$

This vector is represented pictorially below.

Putting the Fourier coefficients just calculated into the Fourier Synthesizer gives a function $\widetilde{f}$. This signal obeys $\widetilde{f}(t) = f(t)$ at all points in the basic interval $(0, 2\pi)$ because f is continuous there. Also, $\widetilde{f}$ has the jump-averaging property shared by all signals emerging from the Fourier Synthesizer. Hence $\widetilde{f}(2\pi n) = \pi$ for all $n \in \mathbb{Z}$.



Partial Sums; Convergence. Given a vector of Fourier coefficients $(\ldots, c_{-1}, c_0, c_1, \ldots)$, using just the middle $2N+1$ terms produces this smooth periodic signal:

$$S_N(t) = \sum_{k=-N}^{N} c_k e^{ik\omega t}.$$

Call this a **partial sum,** in contrast to the full series

$$\widetilde{f}(t) = \sum_{k=-\infty}^{\infty} c_k e^{ik\omega t}.$$

By definition, for each real t (each treated separately),

$$\widetilde{f}(t) = \lim_{N\to\infty} \sum_{k=-N}^{N} c_k e^{ik\omega t} = \lim_{N\to\infty} S_N(t).$$

The limit process preserves periodicity, but not smoothness: the function $\widetilde{f}$ may have corners or even jump discontinuities!

Jump-Averaging. We call a signal h *jump-averaging* if it obeys

$$h(t) = \frac{h(t-) + h(t+)}{2} \qquad \text{for all } t \in \mathbb{R}.$$

On the right we are using the notation

$$h(t-) = \lim_{\substack{x\to t\\ x<t}} h(x), \qquad h(t+) = \lim_{\substack{x\to t\\ x>t}} h(x).$$

At any point t where h is continuous, we get $h(t-) = h(t) = h(t+)$, so the jump-averaging property is satisfied. But if h is discontinuous at t, the property holds only when the function value lies exactly halfway between the values one would predict using one-sided limits. This is relevant because (under reasonable hypotheses on the input $\mathbf{c}$)

All functions produced by the Fourier Synthesizer are jump-averaging.

Recall the proposed process

$$\boxed{\begin{matrix}\text{Signal}\\ f(t)\end{matrix}} \to \boxed{\begin{matrix}\text{FOURIER}\\ \text{ANALYZER}\end{matrix}} \to \boxed{\begin{matrix}\text{sequence}\\ \mathbf{c} = (c_k)\end{matrix}} \to \boxed{\begin{matrix}\text{FOURIER}\\ \text{SYNTHESIZER}\end{matrix}} \to \boxed{\begin{matrix}\text{Signal}\\ \widetilde{f}(t) = \displaystyle\sum_{k=-\infty}^{\infty} c_k e^{ik\omega t}\end{matrix}}.$$

Our main theorem says that this process "cleans up" the input signal f by producing a jump-averaging version $\widetilde{f}$.

Theorem (Pointwise Convergence). *Suppose $f\colon \mathbb{R} \to \mathbb{C}$ is T-periodic and both f, f' are piecewise continuous. Then $\widetilde{f}(t)$ is also T-periodic, and*

$$\widetilde{f}(t) = \frac{f(t-) + f(t+)}{2} \qquad \text{for all } t \in \mathbb{R}.$$

(In particular, $\widetilde{f}(t) = f(t)$ whenever t is a continuity point of f.)

The theorem lets us graph the Fourier reconstruction $\widetilde{f}$ for a given signal f without any calculation.

(a) Pick any open interval of length T. Graph the continuous part of f there. (Leave holes in graph at discontinuities.)

(b) Make a T-periodic extension.

(c) Fill holes by averaging jumps.

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Example. Suppose

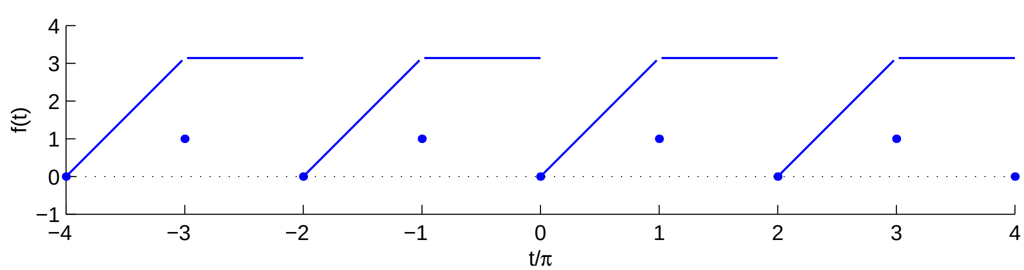
$$f(t) = \begin{cases} t, & \text{for } 0 \le t < \pi, \\ 1, & \text{for } t = \pi, \\ \pi, & \text{for } \pi < t < 2\pi, \end{cases}$$

and $f(t) = f(t + 2\pi)$ for all t. Let $\widetilde{f}(t) = \displaystyle\sum_{k=-\infty}^{\infty} c_k e^{ikt}$ be the Fourier series for f.

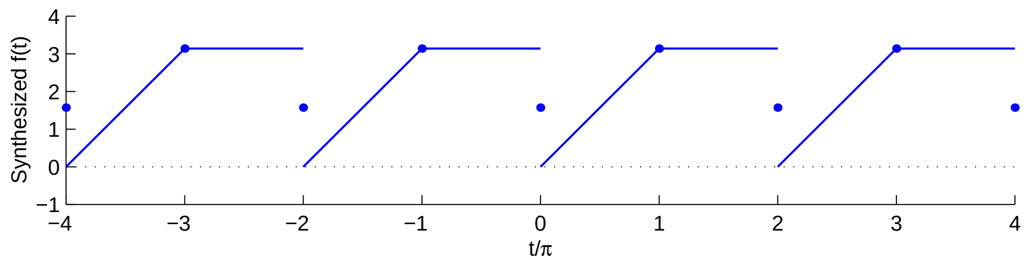
(a) Graph f and $\widetilde{f}$ on interval $[-4\pi, 4\pi]$.

(b) Find the Fourier coefficients c_k.

Solution. (a) Here is a sketch of the given periodic function f. Function values at discontinuity points are shown as extra-large dots:



Here is the Fourier synthesizer output function $\widetilde{f}$. Its values at continuity points match the values of f exactly, but jump-averaging repositions the function values at the discontinuity points:



(b) Break up the integration interval $[0, T]$ into subintervals on which the given function f has a simple form, but keep the same factor $1/T$ out front:

$$\begin{aligned} c_m &= \frac{1}{T}\int_{t=0}^{T} f(t)e^{-imt}\,dt \\ &= \frac{1}{2\pi}\left[\int_0^{\pi} te^{-imt}\,dt + \int_{\pi}^{2\pi} \pi e^{-imt}\,dt\right] \end{aligned}$$

Students can work out the details. They should get

$$\begin{aligned} m = 0 &\implies c_0 = \frac{3\pi}{4}, \\ m \neq 0 &\implies c_m = \left(\frac{(-1)^m - 1}{2\pi m^2}\right) + \frac{i}{2m}. \end{aligned}$$

F. Other Convergence Concepts

Suppose a T-periodic signal f is given; let $\omega = 2\pi/T$. Assume f is jump-averaging, so $f \equiv \widetilde{f}$. Recall the partial sums

$$S_N(t) = \sum_{k=-N}^{N} c_k e^{ik\omega t},$$

built using the Fourier coefficients

$$c_k = \frac{1}{T}\int_0^T f(t)e^{-ik\omega t}\,dt, \qquad k \in \mathbb{Z}.$$

Question: How can we interpret a statement like, "$S_N \approx f$ for N large"?

Subtlety: To measure the distance between complex constants z and w, the single number $|w - z|$ does the job: it's zero when $w = z$ and small when $w \approx z$. But when we discuss the discrepancy between *signals* S_N and f, the analogous quantity $|S_N(t) - f(t)|$ is time-varying. This leads to several *non-equivalent* possible ways to capture the informal idea "$S_N - f$ is small" in a precise mathematical statement.

Three Convergence Concepts: Each of these says a different kind of discrepancy between S_N and f can be made arbitrarily small by summing enough terms.

1. **Pointwise Convergence:** $$\lim_{N\to\infty} |S_N(t) - f(t)| = 0 \quad \text{for each } t \in \mathbb{R}.$$

   This tests discrepancy point-by-point. The number of terms considered "enough" may differ from one point to another.

2. **Uniform Convergence:** $$\lim_{N\to\infty} \left[\max_{t\in\mathbb{R}} |S_N(t) - f(t)|\right] = 0.$$

   This tests the maximum (worst-case) discrepancy. A large number of terms will make the discrepancy small everywhere at once.

3. **Mean-Square Convergence:** $$\lim_{N\to\infty} \left[\frac{1}{T}\int_0^T |S_N(t) - f(t)|^2\, dt\right] = 0.$$

   This tests the average squared-discrepancy over one period.

Discussion. Mean-square convergence is the most common. It happens whenever both f and f' are piecewise continuous, and also in some more general cases.

Pointwise convergence was discussed in the previous section. Whenever both f and f' are piecewise continuous, pointwise convergence is guaranteed.

Uniform convergence is the nicest kind of the three, but it's not always present. We will get uniform convergence whenever f is continuous on $\mathbb{R}$ and f' is piecewise continuous on $\mathbb{R}$, but if f has discontinuities uniform convergence will fail. This is the Gibbs-Wilbraham Phenomenon—see textbook. (Optional: *The condition $\sum_{k=-\infty}^{\infty} |c_k| < +\infty$ guarantees uniform convergence,* by the Weierstrass M-test.)

Convergence Rate; Coefficient Estimates. (Out of time ... maybe later.)

G. Evaluation Tricks and Special Forms

Standard Setup. Let f be T-periodic with f, f' piecewise continuous. Define $\omega = 2\pi/T$ and $\ell = T/2$. Continue with these assumptions throughout this section.

Exact Values. For every integer k,

$$\begin{aligned} e^{ik\pi} &= (-1)^k, & e^{i(2k-1)\pi/2} &= (-1)^{k+1} i, \\ \cos(k\pi) &= (-1)^k, & \cos((2k-1)\pi/2) &= 0, \\ \sin(k\pi) &= 0, & \sin((2k-1)\pi/2) &= (-1)^{k+1}. \end{aligned}$$

Please apply these simplifications whenever you solve problems. It is not required to simplify the expressions $\sin(k\frac{\pi}{2})$ and $\cos(k\frac{\pi}{2})$: these generate four-step sequences for which a concise expression is not immediately available.

The Periodicity Trick. When $\theta = 0$, the Fourier coefficients for f are

$$c_k = \frac{1}{T}\int_\theta^{\theta+T} f(t)e^{-ik\omega t}\,dt, \qquad k \in \mathbb{Z}. \tag{$*$}$$

But in fact, $(*)$ remains true for any and all real θ, because the function integrated here is T-periodic. A popular alternative to $\theta = 0$ is $\theta = -\ell$: it gives

$$c_k = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(t)e^{-ikt/\ell}\,dt, \qquad k \in \mathbb{Z}. \tag{$**$}$$

Trigonometric Form. Adding terms in pairs lets us rearrange the usual series as follows:

$$\widetilde{f}(t) = \sum_{k=-\infty}^{\infty} c_k e^{ik\omega t} = c_0 + \sum_{k=1}^{\infty}\left[c_k e^{ik\omega t} + c_{(-k)}e^{-ik\omega t}\right].$$

In the second form, we notice

$$\begin{aligned} c_k e^{ik\omega t} + c_{(-k)}e^{-ik\omega t} &= c_k\left[\cos(k\omega t) + i\sin(k\omega t)\right] + c_{(-k)}\left[\cos(k\omega t) - i\sin(k\omega t)\right] \\ &= (c_k + c_{-k})\cos(k\omega t) + i\left(c_k - c_{(-k)}\right)\sin(k\omega t), \end{aligned}$$

so by defining $a_k = c_k + c_{-k}$ and $b_k = i(c_k - c_{-k})$, we get the form

$$\widetilde{f}(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k\cos(k\omega t) + b_k\sin(k\omega t)\right).$$

By definition,

$$\begin{aligned} a_k &= \frac{1}{T}\int_0^T f(t)e^{-ik\omega t}\,dt + \frac{1}{T}\int_0^T f(t)e^{-i(-k)\omega t}\,dt \\ &= \frac{1}{T}\int_0^T f(t)\left[e^{ik\omega t} + e^{-ik\omega t}\right]\,dt = \frac{2}{T}\int_0^T f(t)\cos(k\omega t)\,dt, \end{aligned}$$

$$\begin{aligned} b_k &= \frac{i}{T}\int_0^T f(t)e^{-ik\omega t}\,dt - \frac{i}{T}\int_0^T f(t)e^{-i(-k)\omega t}\,dt \\ &= \frac{-i}{T}\int_0^T f(t)\left[e^{ik\omega t} - e^{-ik\omega t}\right]\,dt = \frac{2}{T}\int_0^T f(t)\sin(k\omega t)\,dt, \end{aligned}$$

Using the periodicity trick above gives this summary statement:

Trigonometric Fourier Series

$$\widetilde{f}(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k\cos\left(\frac{k\pi t}{\ell}\right) + b_k\sin\left(\frac{k\pi t}{\ell}\right)\right)$$

$$a_k = \frac{1}{\ell}\int_{-\ell}^{\ell} f(t)\cos\left(\frac{k\pi t}{\ell}\right)\,dt, \qquad k = 0, 1, 2, \ldots,$$

$$b_k = \frac{1}{\ell}\int_{-\ell}^{\ell} f(t)\sin\left(\frac{k\pi t}{\ell}\right)\,dt, \qquad k = 1, 2, \ldots.$$

Important: This is the same function $\widetilde{f}$ as before: only its algebraic appearance has changed! Complex-valued signals f are still eligible (they produce complex a_k, b_k), but it's clear that if f is real-valued then all a_k, b_k will be real-valued as well. Moreover, simple algebra gives

$$\left.\begin{aligned} a_k &= c_k + c_{-k}, \\ b_k &= i(c_k - c_{-k}) \end{aligned}\right\} \iff \left\{\begin{aligned} c_k &= \tfrac{1}{2}(a_k - ib_k) \\ c_{-k} &= \tfrac{1}{2}(a_k + ib_k). \end{aligned}\right. \tag{$**$}$$

We deduce an equivalence we have mentioned before:

$$\boxed{\widetilde{f} \text{ is real-valued} \iff a_k, b_k \in \mathbb{R} \text{ for all } k \iff c_{(-k)} = \overline{c_k} \text{ for all } k.}$$

Textbook Support. Read Boyce/DiPrima Sections 10.2–10.4, using line ($**$) to translate our preferred complex-exponential form into their preferred trigonometric form.

Average Value Trick. In most examples, the constant term in the Fourier series (c_0 in the complex form, $a_0/2$ in the trig form) must be calculated separately from all the rest. It's

$$\frac{a_0}{2} = c_0 = \frac{1}{T}\int_0^T f(t)\,dt,$$

the average value of f over a single period. (Often you can find it geometrically.) In electrical terminology, this number is the "DC component", "DC offset" or "bias" in the signal f.

Odd Symmetry: Simplifications and Extensions. Suppose our 2ℓ-periodic function f is also **odd,** which means $f(-t) = -f(t)$ for all t. In this case we get

$$a_k = \frac{1}{\ell}\int_{-\ell}^{\ell} f(t)\cos\left(\frac{k\pi t}{\ell}\right)\,dt = 0 \quad \text{[integrand is odd]},$$

$$b_k = \frac{1}{\ell}\int_{-\ell}^{\ell} f(t)\sin\left(\frac{k\pi t}{\ell}\right)\,dt = \frac{2}{\ell}\int_0^{\ell} f(t)\sin\left(\frac{k\pi t}{\ell}\right)\,dt \quad \text{[integrand is even]}.$$

The Fourier series takes this simpler form, where *all even components are removed*:

Fourier Sine Series (FSS) for $f\colon [0,\ell] \to \mathbb{C}$
$\displaystyle \widetilde{f}(t) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi t}{\ell}\right)$ $\displaystyle b_k = \frac{2}{\ell}\int_0^{\ell} f(t)\sin\left(\frac{k\pi t}{\ell}\right)\,dt, \qquad k = 1, 2, \ldots.$

Note, in particular, this fact we have mentioned before:

$$\boxed{\widetilde{f} \text{ is odd} \iff a_k = 0 \text{ for all } k \iff c_{(-k)} = -c_k \text{ for all } k.}$$

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Now for a significant extension. Imagine applying the boxed formulas to some arbitrary f (perhaps not even periodic) whose domain includes the interval $[0, \ell]$. Boldly use the integrals above to define some constants b_k, and plug those in the series above to define a function $\widetilde{f}$. Call $\widetilde{f}$ the Fourier Sine Series (FSS) for f. What does $\widetilde{f}$ look like? Just look at the series and remember the main convergence theorem:

- ☐ $\widetilde{f}(t) = f(t)$ for each continuity point t of f in $(0, \ell)$;
- ☐ $\widetilde{f}$ is an odd function;
- ☐ $\widetilde{f}$ is 2ℓ-periodic;
- ☐ $\widetilde{f}$ is jump-averaging.

There is enough info here to graph $\widetilde{f}$ exactly: it's the Fourier series for the jump-averaging, 2ℓ-periodic, odd extension of f.

Even Symmetry: Simplifications and Extensions. Suppose our 2ℓ-periodic function f is also **even,** which means $f(-t) = f(t)$ for all t. In this case we get

$$a_k = \frac{1}{\ell}\int_{-\ell}^{\ell} f(t)\cos\left(\frac{k\pi t}{\ell}\right)\,dt = \frac{2}{\ell}\int_0^{\ell} f(t)\cos\left(\frac{k\pi t}{\ell}\right)\,dt \quad \text{[integrand is even]},$$

$$b_k = \frac{1}{\ell}\int_{-\ell}^{\ell} f(t)\sin\left(\frac{k\pi t}{\ell}\right)\,dt = 0 \quad \text{[integrand is odd]}.$$

The Fourier series takes this simpler form, where *all odd components are removed*:

Fourier Cosine Series (FCS) for $f\colon [0, \ell] \to \mathbb{C}$

$$\widetilde{f}(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi t}{\ell}\right)$$

$$a_k = \frac{2}{\ell}\int_0^{\ell} f(t)\cos\left(\frac{k\pi t}{\ell}\right)\,dt, \qquad k = \; 1, 2, \ldots.$$

Note, in particular, this fact we have mentioned before:

$$\boxed{\widetilde{f} \text{ is even} \iff b_k = 0 \text{ for all } k \iff c_{(-k)} = c_k \text{ for all } k.}$$

Now for a significant extension. Imagine applying the boxed formulas to some arbitrary f (perhaps not even periodic) whose domain includes the interval $[0, \ell]$. Boldly use the integrals above to define some constants a_k, and plug those in the series above to define a function $\widetilde{f}$. Call $\widetilde{f}$ the Fourier Cosine Series (FCS) for f. What does $\widetilde{f}$ look like? Just look at the series and remember the main convergence theorem:

- ☐ $\widetilde{f}(t) = f(t)$ for each continuity point t of f in $(0, \ell)$;

- □ $\widetilde{f}$ is an even function;
- □ $\widetilde{f}$ is 2ℓ-periodic;
- □ $\widetilde{f}$ is jump-averaging.

There is enough info here to graph $\widetilde{f}$ exactly: it's the Fourier series for the jump-averaging, 2ℓ-periodic, even extension of f.

Coefficient-Matching Trick. Let $f\colon [0,5] \to \mathbb{R}$ be defined by

$$f(x) = 3\sin(3\pi x) - 7\sin(7\pi x) + 101\sin(8\pi x).$$

Find the Fourier series for f.

Solution. The given expression for f is continuous and 10-periodic as written so $\widetilde{f} = f$. Hence

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{5}\right)$$

for some b_n. Simply comparing this FSS form to the rearrangement

$$f(x) = 3\sin\left(\frac{15\pi x}{5}\right) - 7\sin\left(\frac{35\pi x}{5}\right) + 101\sin\left(\frac{40\pi x}{5}\right),$$

reveals $b_{15} = 3$, $b_{35} = -7$, $b_{40} = 101$, all other $b_n = 0$. ////

See also Problem Set 1.

H. FSS Examples (for student practice—not shown in class)

Example 1. *Find the Fourier Sine Series on $0 < x < \pi$ for this function:*

$$f(x) = \begin{cases} 1, & \text{if } x \neq \pi/2, \\ 0, & \text{if } x = \pi/2. \end{cases}$$

Solution. Here $\ell = \pi$—the simplest case. The Fourier Sine Analyzer gives out same sequence as it would for constant function 1, namely

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} 1\sin(nx)\,dx = \frac{2}{\pi}\left[\frac{-\cos(nx)}{n}\right]_0^{\pi} = \frac{2}{n\pi}\left[1-\cos(n\pi)\right] \\ &= \frac{2}{\pi}\left(\frac{1-(-1)^n}{n}\right) = \begin{cases} 4/(n\pi), & \text{if } n \text{ odd}, \\ 0, & \text{if } n \text{ even}. \end{cases} \end{aligned}$$

Stick this into Fourier Sine Synthesizer to get

$$\begin{aligned} \widetilde{f}(x) &= \sum_{n=1}^{\infty} b_n \sin(nx) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n}\right)\sin(nx) \\ &= \frac{4}{\pi}\sum_{n=1,3,5,\ldots} \frac{1}{n}\sin(nx) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k-1}\sin\left((2k-1)x\right). \end{aligned}$$

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Example 2. *Let $f(x) = |x|$. Find the Fourier Sine Series on $0 < x < \pi$.*

Solution. All values outside the interval $0 < x < \pi$ are ignored. In this interval, $f(x) = x$, so

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^\pi x\sin(nx)\,dx \\ &= \frac{2}{\pi}\left[x\left(\frac{-\cos(nx)}{n}\right)\Big|_{x=0}^{\pi} - \int_0^\pi \left(\frac{-\cos(nx)}{n}\right)\,dx\right] \\ &= \frac{2}{\pi}\left[-\frac{\pi}{n}\cos(n\pi) + \frac{\sin(nx)}{n^2}\Big|_{x=0}^{\pi}\right] \\ &= -\frac{2}{n}(-1)^n. \end{aligned}$$

////

Example 3. *Find the FSS on $[0,2]$ and sketch $\widetilde{f}$, given*

$$f(x) = \begin{cases} 2, & \text{if } x = 0, \\ 1-x, & \text{if } 0 < x < 1, \\ 1, & \text{if } x = 1, \\ 0, & \text{if } 1 < x \le 2. \end{cases}$$

Solution. Splitting and IBP give

$$b_n = \begin{cases} \dfrac{2}{n\pi} - 0, & \text{if } n \text{ even}, \\ \dfrac{2}{n\pi} - \dfrac{4}{n^2\pi^2}, & \text{if } n = 1, 5, 9, 13, \ldots, \\ \dfrac{2}{n\pi} + \dfrac{4}{n^2\pi^2}, & \text{if } n = 3, 7, 11, 15, \ldots. \end{cases}$$

Reconstruct sketch as noted above. ////

Example 4. *Give FSS for each of these functions on interval $0 < x < \pi$:*

$$f(x) = 1, \qquad g(x) = |\sin x|, \qquad h(x) = |\cos x|.$$

Solution. $f(x) = 1$ done before. On interval $[0,\pi]$, we have $g(x) = \sin(x)$. This is a FSS already, with $b_1 = 1$ and all other $b_n = 0$, giving $\widetilde{g}(x) = \sin(x)$. Compare graphs of g and $\widetilde{g}$. Function $h(x)$ takes work. Use some trig identities. ////

I. Parseval's Equation

The key to this whole story is the orthogonality relation we first saw at the beginning of Section D (here $\omega = 2\pi/T$):

$$\int_\theta^{\theta+T} e^{ik\omega t}\, e^{-ikm\omega t}\,dt = \begin{cases} 0, & \text{if } k \ne m, \\ T, & \text{if } k = m. \end{cases}$$

Recall our notation $S_N(t) = \sum_{k=-N}^{N} c_k e^{ik\omega t}$. Orthogonality gives

$$\begin{aligned}
\int_0^T |S_N(t)|^2\,dt &= \int_0^T \left(\sum_{k=-N}^{N} c_k e^{ik\omega t}\right)\overline{\left(\sum_{m=-N}^{N} c_m e^{im\omega t}\right)}\,dt \\
&= \sum_{k=-N}^{N}\sum_{m=-N}^{N}\int_0^T c_k e^{ik\omega t}\overline{c_m}e^{-im\omega t}\,dt \\
&= \sum_{k=-N}^{N} T c_k \overline{c_k} \\
&= T\sum_{k=-N}^{N} |c_k|^2.
\end{aligned}$$

Now consider: $\widetilde{f}$ and f behave identically for purposes of integration on $[\theta, \theta+T]$, so

$$\begin{aligned}
\int_0^T |f(t) - S_N(t)|^2\,dt &= \int_0^T |f(t)|^2 + |S_N(t)|^2\,dt - 2\,\mathrm{Re}\left(\int_0^T f(t)\overline{S_N(t)}\,dt\right) \\
&= \int_0^T |f(t)|^2 + |S_N(t)|^2\,dt - 2\,\mathrm{Re}\left(\sum_{k=-N}^{N}\overline{c_k}\int_0^T f(t)e^{-ik\omega t}\,dt\right) \\
&= \int_0^T |f(t)|^2 + |S_N(t)|^2\,dt - 2T\,\mathrm{Re}\left(\sum_{k=-N}^{N}\overline{c_k}\,Tc_k\right) \\
&= \int_0^T |f(t)|^2\,dt - T\sum_{k=-N}^{N}|c_k|^2.
\end{aligned}$$

Now send $N \to \infty$: LHS tends to 0 (mean-square convergence), leaving

$$\boxed{\frac{1}{T}\int_0^T |f(t)|^2\,dt = \sum_{k=-\infty}^{\infty}|c_k|^2.}$$

This is **Parseval's Equation.** It says that the RMS value for one period of signal f is precisely captured by the generalized Pythagorean length of the vector of Fourier coefficients. This is a significant relationship between the time-domain (where the real signal lives) and the frequency-domain (where we look at the spectral components). Understanding signals in both domains gives engineers great power and flexibility for imagination and design.

Variations. The trigonometric form of Parseval's identity says

$$\widetilde{f}(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k\cos(k\pi t/\ell) + b_k\sin(k\pi t/\ell)\right)$$

$$\implies \frac{1}{\ell}\int_{-\ell}^{\ell}\left|\widetilde{f}(t)\right|^2\,dt = \frac{a_0^2}{2} + \sum_{k=1}^{\infty}\left(|a_k|^2 + |b_k|^2\right).$$

Other special forms include

$$\text{FSS:} \qquad \widetilde{f}(t) = \sum_{k=1}^{\infty} b_k \sin(k\pi t/\ell) \qquad \Rightarrow \qquad \frac{2}{\ell}\int_0^{\ell} \left|\widetilde{f}(t)\right|^2 dt = \sum_{k=1}^{\infty} |b_k|^2,$$

$$\text{FCS:} \qquad \widetilde{f}(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\pi t/\ell) \qquad \Rightarrow \qquad \frac{2}{\ell}\int_0^{\ell} \left|\widetilde{f}(t)\right|^2 dt = \frac{a_0^2}{2} + \sum_{k=1}^{\infty} |a_k|^2.$$

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Example. In Section H, Example 1, we have the FSS identity

$$1 = \sum_{k=1,3,5,\ldots} \frac{4}{k\pi}\sin(kx), \qquad 0 < x < \pi.$$

Here $\ell = \pi$, and the FSS form of Parseval’s identity gives

$$\frac{2}{\pi}\int_0^{\pi} |1|^2\,dt = \sum_{k=1,3,5,\ldots} \frac{16}{k^2\pi^2}.$$

Evaluating the integral and rearranging gives

$$\frac{\pi^2}{8} = \sum_{k=1,3,5,\ldots} \frac{1}{k^2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots.$$

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The derivation above produced a formula that is also useful when N is finite:

$$\frac{1}{T}\int_0^T |f(t) - S_N(t)|^2\,dt = \sum_{|k|>N} |c_k|^2.$$

This equation shows how the RMS error in the approximation $f \approx S_N$ is exactly accounted for by the high-frequency energy that is neglected when we use the partial sum to replace the full series.