MAASS FORMS AND L-FUNCTIONS

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Abstract. Lectures given at the summer school "Automorphic Forms and L-Functions: Computational Aspects" held at the CRM (Montreal) in June–July 2009. Notes by Lior Silberman (`lior@math.ubc.ca`) who responsible for all errors and omissions. Corrections welcome.

Goal: compute L-functions associated to automorphic forms on GL_2/F for F totally real.

Part 1. Fun with the FFT

Using the FFT one can efficiently compute many L-values using algorithms for computing a single value.

(1) $L(\frac{1}{2}+it)$ for many values of t (L an L-function)
(2) $L(s;\chi)$ for many values of χ (current work of student D. Platt)

Using this can test the RH.

1. Intro

Recall the Riemann-Siegel formula ("approximate FE"): let $Z(t)$ be the Riemann zetafunction rotated and rescaled to be real on the real axis. Then:

$$Z(t) = 2\Re \sum_{n \le \sqrt{t/2\pi}} n^{\frac{1}{2}} e^{i(\theta(t) - t\log n)} + O(t^{-1/4}),$$

$$\theta(t) = \frac{t}{2}\log\frac{t}{2\pi e} - \frac{\pi}{8} + O(t^{-1}).$$

Remark 1. In practice, need to refine the formula, taking more terms and getting better control on the error term.

Problem 2. Evaluate this for (many) values of t in an interval $[t_0 - h, t_0 + h]$.

- If $h = O(t_0^{1/2})$, $\sqrt{t/2\pi}$ differs from $\sqrt{t_0/2\pi}$ by a bounded amount, so at the cost of increasing the error term by a constant can have the number of terms fixed.
- Writing $t = t_0 + \delta$ and taking a Taylor expansion in $\theta(t) = \frac{t}{2}\log\frac{t_0}{2\pi} - \frac{\pi}{8} + O(t_0^{-1}) + \frac{t}{2}\log\frac{1+\delta/t_0}{e}$ we have:

$$\theta(t) \approx \frac{t}{2}\log\frac{t_0}{2\pi} - \frac{\pi}{8} - \frac{t}{2} + O\left(\frac{h^2+1}{t_0}\right).$$

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Thus if $h = O(t_0^{1/4})$ we have:

$$Z(t) = 2\Re\left[e^{i\left(\frac{t_0}{2}-\frac{\pi}{8}\right)} \sum_{n\le\sqrt{t_0/2\pi}} n^{-\frac{1}{2}} e^{it\log\frac{\sqrt{t_0/2\pi}}{n}}\right] + O\left(t_0^{-1/4}\right).$$

Now we have $Z(t)$ as (roughly) a "band-limited" function – its Fourier Transform has bounded support: the frequencies range between zero (when n is largest) and $\frac{1}{2}\log\frac{t_0}{2\pi}$ (when $n=1$)

- The approach works for general L-functions with $\frac{t_0}{2\pi}$ replaced, in general, with the *analytic conductor*.

2. The Fourier Transform

For $f \in \mathcal{S}(\mathbb{R})$ set

$$\hat{f}(\nu) = \int_{\mathbb{R}} f(t)e(-\nu t)dt\,.$$

Then (Fourier inversion)

$$f(t) = \int_{\mathbb{R}} \hat{f}(\nu)e(\nu x)d\nu\,.$$

Function	Fourier transform
$f(t)$	$\hat{f}(\nu)$
$t \mapsto f(at)$	$\nu \mapsto \frac{1}{a}\hat{f}(\frac{\nu}{a})$
$(f*g)(t)$	$\hat{f}(\nu)\hat{g}(\nu)$
$\delta(t)$	1
$c(t) = \sum_{n\in\mathbb{Z}}\delta(t-n)$	$\hat{c}(\nu) = c(\nu)^1$
$f(t) = \mathrm{sinc}(\pi t) = \frac{\sin(\pi t)}{\pi t}$	$\hat{f}(\nu) = \chi_{[-\frac{1}{2},\frac{1}{2}]}(\nu)$

Finally, if $\hat{f}(\nu)$ is supported in $(-1,1)$, then $\hat{f}(\nu) = \left(\sum_{n\in\mathbb{Z}}\hat{f}(\nu-n)\right)\cdot\chi_{[-1/2,1/2]}(\nu)$. Fourier inversion and the Poisson Summation Formula give:

$$f(t) = \sum_{n\in\mathbb{Z}} f(n)\,\mathrm{sinc}\left[\pi(t-n)\right]$$

if f has bandwidth at most $\frac{1}{2}$. More generally, we obtain:

Proposition 3. *("Sampling Theorem") If $f(t)$ has bandwidth at most B (that is, $\hat{f}$ is supported in $[-B,B]$), then f can be recovered by sampling at the rate $2B$ (the* Nyquist frequency*), that is at $\frac{1}{2B}\mathbb{Z}$:*

$$f(\frac{t}{2B}) = \sum_{n\in\mathbb{Z}} f\left(\frac{n}{2B}\right)\mathrm{sinc}\left[\pi(t-n)\right]\,.$$

Returning to $\zeta(t)$, should be enough to sample with frequency $\frac{1}{2\pi}\log\frac{t}{2\pi}$.

3. L-functions

We think of a degree d Euler product

$$L(s) = \prod_p L_p(s) = \prod_p\prod_{j=1}^{d}\left(1-\alpha_p^{(j)}p^{-s}\right)^{-1}\,.$$

We normalize the coefficients so that the functional equation relates $L(s)$ and $L(1-s)$. In this normalization we usually have $\left|\alpha_p^{(j)}\right| \ll p^{\vartheta}$ for some $\vartheta \leq \frac{1}{2}$ (unitarity). In particular, we have:

Conjecture 4. *(Generalized Ramanujan Conjecture)* $\left|\alpha_p^{(j)}\right| \leq 1$.

Theorem 5. *(Kim-Sarnak) For Maass forms one may take* $\vartheta = \frac{7}{64}$.

Definition 6. Let $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)$, $\Gamma_{\mathbb{C}} = 2(2\pi)^{-s}\Gamma(s) = \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1)$. Then:

$$\Gamma_{\mathbb{R}}(s) = \int_{\mathbb{R}^\times} e^{-\pi t^2}\,|t|^s\,\frac{dt}{|t|}\,.$$

For our L-function we assume that we have a factor at infinity

$$L_\infty(s) = \prod_{j=1}^{d} \Gamma_{\mathbb{R}}(s+\mu_j)\,.$$

The GRC here reads $\Re(\mu_j) = 0$; we'll assume $|\Re(\mu_j)| \leq \theta < \frac{1}{2}$.

Then $\Lambda(s) = L_\infty(s)L(s)$ satisfies a functional equation

$$\Lambda(s) = wN^{\frac{1}{2}-s}\overline{\Lambda(1-\bar{s})}\,.$$

The analytic conductor is then $C(s) = N\prod_{j=1}^{d}\frac{s+\mu_j}{2\pi}$.

4. The decay in the Fourier transform of the L-function

Instead write $\gamma(s) = w^{-\frac{1}{2}}N^{\frac{1}{2}(s-1/2)}L_\infty(s)$ so the functional equation of $\Lambda(s) = \gamma(s)L(s)$ is $\Lambda(s) = \overline{\Lambda(1-\bar{s})}$, in particular $\Lambda(\frac{1}{2}+it) = \overline{\Lambda(\frac{1}{2}+it)}$. In other words, this function is real on the "real axis".

Since $\left|\gamma(\frac{1}{2}+it)\right| \sim e^{-\frac{\pi d}{4}|t|}$, natural to consider $\Lambda\left(\frac{1}{2}+it\right)e^{\frac{\pi d}{4}|t|}$, which is of at most polynomial growth. Would also like to restrict the function to the interval $[t_0-h, t_0+h]$ with optimal restriction on the Fourier transform side. In fact we consider

$$f(t) = \Lambda\left(\frac{1}{2}+it\right)e^{-\frac{\pi d}{4}t}e^{-(t-t_0)^2/2h^2}\,.$$

Use Gaussian since it is an entire function. It also has rapid decay in vertical strips, allowing us to shift contours: set

$$\hat{f}(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(t)e^{-ixt}dt = \frac{1}{2\pi i}\int_{\Re(s)=\frac{1}{2}} \Lambda(s)e^{-\left(x+\frac{i\pi d}{4}\right)\left(s-\frac{1}{2}\right)+\left(s-\frac{1}{2}-it_0\right)^2/2h^2}ds\,.$$

Can shift contours to $\Re(s) = \sigma$. Assuming no poles, get:

$$\hat{f}(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\Lambda(\sigma+it)\exp\left\{-\left(x+\frac{i\pi d}{4}\right)\left(\sigma-\frac{1}{2}+it\right)+\left(\sigma-\frac{1}{2}+i(t-t_0)\right)^2/2h^2\right\}dt\,,$$

So

$$\left|\hat{f}(x)\right| \leq \frac{1}{2\pi}\int_{\mathbb{R}} N^{\frac{1}{2}(\sigma-\frac{1}{2})}\prod_j\left|\Gamma_{\mathbb{R}}\left(\sigma+it+\mu_j\right)\right|\left|L\left(\sigma+it\right)\right|e^{\frac{\pi d}{4}t}e^{-x(\sigma-\frac{1}{2})}\exp\left\{\frac{(\sigma-\frac{1}{2})^2-(t-t_0)^2}{2h^2}\right\}dt\,.$$

Using $\log \Gamma_{\mathbb{R}}(s) = \frac{s-1}{2} \log \frac{s-1}{2\pi e} + \frac{1}{2} \log 2 + O\left(\frac{1}{|s|}\right)$, $\Re \log \Gamma_{\mathbb{R}}\left(\sigma + it + \mu_j\right) = \frac{\sigma - \frac{1}{2}}{2} \log \frac{t}{2\pi} - \frac{\pi t}{4} + O(1/t)$. Thus

$$\hat{f}(x) \ll \int dt\, |L(\sigma + it)| \exp\left\{ (\sigma - \frac{1}{2}) \left(\log \sqrt{N(t/2\pi)^r} - x \right) + \frac{(\sigma - \frac{1}{2})^2 - (t - t_0)^2}{2h^2} \right\}.$$

Thus obtain exponential decay for x large. The bound towards Ramanujan assumed before gives $|L(\sigma + it)| \ll \left(\zeta(\sigma - \theta)\right)^r$ uniformly. Note that inside the log can replace t with t_0, to obtain the analytic conductor $Q = Q(\frac{1}{2} + it_0)$. Evaluating the integral gives:

$$\hat{f}(x) \ll h \exp\left\{ (\sigma - \frac{1}{2}) \left(\log \sqrt{Q} - x \right) + \frac{(\sigma - \frac{1}{2})^2}{2h^2} \right\}.$$

Optimizing the choice of σ gives:

$$\hat{f}(x) \ll h \exp\left\{ -\frac{h^2}{2} \left(x - \frac{1}{2} \log Q \right) \right\}.$$

This was all for $x > 0$ but since f is real, $\hat{f}(-x) = \overline{\hat{f}(x)}$, giving us decay in both directions.

5. Evaluating $\Lambda(\frac{1}{2} + it)$ from scratch in $[0, T]$.

We have seen that f is band-limited. We can thus reconstruct f from a small number of samples. We will work directly with

$$f(t) = \Lambda\left(\frac{1}{2} + it\right) e^{\frac{\pi d}{4} \eta t}$$

with $\eta \in [0, 1)$ so that $1 - \eta \sim \frac{1}{T}$. Shifting the contour in the Fourier transform,

$$\hat{f}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} f(t) e^{-\pi x t} dt$$

$$= \frac{1}{2\pi i} \int_{\Re(s)=2} \gamma(s) \left(\sum_{n=1}^{\infty} a_n n^{-s} \right) \exp\left\{ \left(x + \frac{i\pi\eta d}{4} \right) (\frac{1}{2} - s) \right\} ds\,.$$

We are in the region of absolute convergence so we can exchange summation and integration and get:

$$= \sum_{n=1}^{\infty} a_n \frac{1}{2\pi i} \int_{\Re(s)=2} \gamma(s) \exp\left\{ \left(x + \frac{i\pi\eta d}{4} \right) (\frac{1}{2} - s) \right\} n^{-s} ds\,.$$

It follows that

$$\hat{f}(x) = w^{-\frac{1}{2}} \sum_{n=1}^{\infty} \frac{a_n}{\sqrt{n}} G(x + \log \frac{n}{\sqrt{N}})$$

where

$$G(u) = \frac{1}{2\pi i} \int_{\Re(s)=2} \exp\left\{ \left(u + \frac{i\pi\eta d}{4} \right) \left(\frac{1}{2} - s \right) \right\} \prod_{j=1}^{d} \Gamma_{\mathbb{R}}\left(s + \mu_j\right) ds\,.$$

Remark 7. It seems the sum for $\hat{f}(x)$ has length $\sqrt{N}$ but examining the decay of G shows that the correct length is $\sqrt{Q}$ where Q is the analytic conductor.

6. From $\hat{f}$ to f: the DFT

Recall the twisted Poisson summation formula:

$$\sum_{n\in\mathbb{Z}} f(\alpha+nB) = \frac{2\pi}{B}\sum_{k\in\mathbb{Z}} \hat{f}\left(\frac{2\pi k}{B}\right) e\left(\frac{k\alpha}{B}\right).$$

Here we use the latter definition of $\hat{f}$, with e^{-ikx} rather than $e(kx)$.

Chose A such that $q = AB \in \mathbb{Z}_{>0}$ and let $\alpha = \frac{m}{A}$, $m \in \mathbb{Z}$. Then

$$\sum_{n\in\mathbb{Z}} f(\frac{m+nq}{A}) = \frac{2\pi}{B}\sum_{k\in\mathbb{Z}} \hat{f}\left(\frac{2\pi k}{B}\right) e\left(\frac{km}{q}\right).$$

Breaking the k-sum over residue classes modulu q we see:

$$\sum_{n\in\mathbb{Z}} f\left(\frac{m+nq}{A}\right) = \frac{2\pi}{B}\sum_{a(q)} e\left(\frac{am}{q}\right)\sum_{k\in\mathbb{Z}} \hat{f}\left(\frac{2\pi a}{B}+2\pi Ak\right).$$

Writing $F(m) = \sum_{n\in\mathbb{Z}} f\left(\frac{m+nq}{A}\right)$ for the LHS, $\tilde{F}(a) = \sum_{k\in\mathbb{Z}} \hat{f}\left(\frac{2\pi a}{B}+2\pi Ak\right)$ from the RHS, both are functions on $\mathbb{Z}/q\mathbb{Z}$ and we have

$$F(m) = \frac{2\pi}{B}\sum_{a(q)} e\left(\frac{am}{q}\right)\tilde{F}(a).$$

There is an algorithm (the FFT) which calculates all values of $F(m)$ given the values $\tilde{F}(a)$ with $O(q\log q)$ floating-point operations. It is especially efficient if q is a product of small primes.

We now choose A, B so that the sums defining F, $\tilde{F}$ push large values of n, k into the asymptotic regions. We conclude that, in order to evaluate $f(t)$ at the q points $\frac{m}{A}$ it suffices to evaluate $\hat{f}(x)$ on a grid of points with spacing $\frac{2\pi}{B}$.

7. From $G(u)$ to $\hat{f}(x)$: the DFT, again

Recall the formula

$$\hat{f}(x) = w^{-\frac{1}{2}}\sum_{n=1}^{\infty}\frac{a_n}{\sqrt{n}}G(x+\log\frac{n}{\sqrt{N}}).$$

We would like to evaluate this for x belonging to a grid of spacing $\frac{2\pi}{B}$. For this let $\{u_m\} \subset \mathbb{R}$ be a grid of spacing $\frac{2\pi}{NB}$ for some integer N, a refinement of the grid where we are evaluating $\hat{f}(x)$, and assume we have tabulated values for the first K derivatives of G at these grid points. We write $\epsilon = \frac{\pi}{NB}$ for the half-spacing of the grid, I_m for the interval $[u_m-\epsilon, u_m+\epsilon]$

Assume now that $\log\frac{n}{\sqrt{N}} \in [u_m-\epsilon, u_m+\epsilon]$ for some u_m in the grid, and let x be one of the points where we would like to evaluate $\hat{f}(x)$. Then $x+u_m$ is also a point in the finer grid, and we have:

$$G(x+\log\frac{n}{\sqrt{N}}) \approx \sum_{k=0}^{K-1}\frac{G^{(k)}(x+u_m)}{k!}\left(\log\frac{n}{\sqrt{N}}-u_m\right)^k$$

(for the error in this approximation in the next section). Multiplying by $\frac{a_n}{\sqrt{n}}$, summing over n and changing the order of summation we conclude:

$$\hat{f}(x) \approx \sum_m \sum_{n:\log\frac{n}{\sqrt{N}}\in I_m} \frac{a_n}{\sqrt{n}} \sum_{k=0}^{K-1} \frac{G^{(k)}(x+u_m)}{k!} \left(\log\frac{n}{\sqrt{N}} - u_m\right)^k .$$

It follows that

$$\hat{f}(x) = \sum_{k=0}^{K-1} \frac{1}{k!} \sum_m G^{(k)}(x+u_m)\cdot S_m^{(k)}$$

with

$$S_m^{(k)} = \sum_{\log\frac{n}{\sqrt{N}}\in I_m} \frac{a_n}{\sqrt{n}} \left(\log\frac{n}{\sqrt{N}} - u_m\right)^k .$$

Note that:

- The $S_m^{(k)}$ can be calculated efficiently by running over n and for each n identifying to $S_m^{(k)}$ the next contribution belongs to.
- In the next section we discuss evaluating the $G^{(k)}(u_m)$.
- We have written $\hat{f}$ as a sum of K terms, each of which is a convolution over the fine grid u_m of the two sequences $S_m^{(k)}$ and $G^{(k)}(u_m)$. Such convolutions can be efficiently computed by the FFT.

8. Evaluating $G(u)$

By Taylor's Theorem, if u is ϵ-close to u_m,

$$G(u) = \sum_{k<K} \frac{G^{(k)}(u_m)}{k!}(u-u_m)^k + O\left(\max_{|u^*-u|\le\epsilon} \frac{\left|G^{(K)}(u^*)\right|}{K!}\epsilon^K\right) .$$

It is important to note that we can bound the derivatives of $G(u)$ independently of u:

$$G^{(k)}(u) = \frac{1}{2\pi i}\int_{\Re(s)=2} \left(\frac{1}{2}-s\right)^k \exp\left\{\left(u+\frac{i\pi\eta d}{4}\right)\left(\frac{1}{2}-s\right)\right\} \prod_{j=1}^d \Gamma_{\mathbb{R}}\left(s+\mu_j\right) ds ,$$

and shifting the contour back to $\Re(s)=\frac{1}{2}$, we have:

$$\left|G^{(k)}(u)\right| \le \frac{1}{2\pi}\int_{\mathbb{R}} \frac{|t|^k}{k!} e^{\frac{\pi\eta d}{4}t} \prod_j \left|\Gamma_{\mathbb{R}}\left(\frac{1}{2}+it+\mu_j\right)\right| dt .$$

This is small as long as η is close enough to 1: say $\frac{\pi d}{4}(1-\eta)=\frac{1}{T}$.

Finally, evaluate $G(u)$ is by shifting the contour far to the left, picking up contributions from the poles of the gamma factor. One gets an expression of the form:

$$G(u) = \sum_{\text{poles } \rho} P_\rho(u) e^{\left(\frac{1}{2}-\rho\right)u} .$$

Remark 8. It is important to note that this method entails an enormous loss of precision. A good example is the case of $e^{-t} = \frac{1}{2\pi i}\int_{\Re(s)=2}\Gamma(s)t^{-s}ds$. Shifting the contour all the way to $\Re(s)=-\infty$ gives the usual power series $e^{-t}=\sum_{k=0}^\infty \frac{(-1)^k}{k!}t^k$. Here the largest summand is of size about e^t, so to calculate e^t this way requires relative precision of more than $1/e^{2t}$.

Nevertheless in many arithmetic families of L-functions the Γ-factors are basically same for the whole family (say, there are at most finitely many different cases[2]. It then makes sense to tabulate values of $G(u)$ and its derivatives once and for all, even if the evaluations are expensive. As we shall see below it usually suffices to evaluate G on a fine enough grid, but even if off-grid values are desired they can be found by Taylor expansion.

9. A discrete family

The situation is much simpler when evaluating a family of L-functions at a fixed value $s \in \mathbb{C}$. As an example, we shall evaluate the numbers $L(s;\chi)$ as χ varies over the Dirichlet characters $\mod q$,where q is a large integer.

9.1. **Zeroth approach.** Write the approximate functional equation

$$L(s;\chi) = \sum_{n=1}^{\infty} \chi(n) f_1(n,s) + w_\chi \sum_{n=1}^{\infty} \overline{\chi(n)} f_2(n, 1-s)\,,$$

where both functions f_1 and f_2 decay for $n \gg \sqrt{q}$ (since s is fixed the analytic conductor is roughly q). This is the optimal division for evaluating a single $L(s;\chi)$ – we'd need time about $O(\sqrt{q})$ for each of them, so time $O(q^{3/2})$ for all of them.

9.2. **First approach.** For $1 \le a \le q$ write:

$$\tilde{f}_i(a,s) = \sum_{n=1}^{\infty} f_i(a+nq, s)\,.$$

This sum is in fact very short and can thus be evaluated efficiently: since f_i decays over the scale $\sqrt{q}$, taking just the first term usually suffices..

Now, since χ is periodic, we have:

$$L(s,\chi) = \sum_{a=1}^{q} \chi(a) \tilde{f}_1(a;s) + \sum_{a=1}^{q} \overline{\chi(a)} \tilde{f}_2(a;1-s)\,.$$

Furthermore, each of the two sums in the RHS is a DFT in the group $(\mathbb{Z}/q\mathbb{Z})^\times$, and hence can be calculated efficiently in time $O(q \log q)$.

Remark 9. This requires constructing an explicit isomorphism between $(\mathbb{Z}/q\mathbb{Z})^\times$ and a direct product of cyclic groups, but that is easy to construct in time $O(q)$ (for example, if q is prime what is needed is to find a primitive root).

9.3. **Second approach.** In fact, there is no reason to take the balanced approximate FE – it is better to work with the Dirichlet series directly (recall that it converges in the critical strip). Again breaking the sum into residue classes, we

[2]Some examples: Dirichlet L-functions have two cases, even and odd. For holomorphic modular forms the factor only depends on the weight. For Artin L-functions of a particular dimension there are finitely many possibilities.

have:

$$\begin{aligned} L(s;\chi) &= \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \\ &= \sum_{a=1}^{q} \chi(a) \sum_{n=1}^{\infty} \frac{1}{(a+nq)^s} \\ &= q^{-s} \sum_{a=1}^{q} \chi(a) Z\left(\frac{a}{q}, s\right) \end{aligned}$$

where $Z(\alpha, s) = \sum_{n=1}^{\infty} (\alpha+n)^{-s}$ is the Hurwitz zeta-function.

We have thus reduced the problem of evaluating $L(s;\chi)$ to the problem of evaluating $Z(\alpha, s)$ where α ranges over q equally spaced points in $[\frac{1}{q}, 1]$. The latter problem is solved using a Taylor expansion: since $\frac{d}{d\alpha} Z(\alpha, s) = -sZ(\alpha, s+1)$, we have:

$$Z(\alpha+\delta, s) = \sum_{j=0}^{\infty} \binom{s+j-1}{j} Z(\alpha, s+j)(-\delta)^j .$$

Since $Z(\alpha, s+j)$ can be calculated directly when j is large (region of absolute convergence), we can recursively evaluate $Z(\alpha, s)$ over finer and finer meshes in a fixed interval.

- Booker's student David Platt has implemented this in interval arithmetic.

Q. (Rubinstein): Won't these methods (periodize then use FFT) have problems of accumulated round-off error?

A. (Booker): Can handle this by calculating values of $Z(\alpha, s)$ in the table to high precision so that we know they are correct. The DFT itself is calculated in double precision.

10. Application: verifying ERH

10.1. **The Riemann $\zeta(s)$.** Let $N(T)$ denote the number of zeroes in the critical strip up to height T, $\theta(t)$ the accumulated phase of the Gamma factor $\gamma(s)$ ($\Gamma_{\mathbb{R}}(s)$ in this case) between $s = \frac{1}{2}$ and $s = \frac{1}{2} + it$.

Fact 10. *$N(t) \approx \frac{\theta(t)}{\pi} + 1$, and $\theta(t) = \frac{t}{2\pi} \log \frac{t}{2\pi e} + \frac{7}{8} + o(1)$.*

It is natural to work with $S(t) = N(t) - \left(\frac{\theta(t)}{\pi} + 1\right)$.

The zeroes of $\zeta(s)$ are found by considering the function $Z(t) = \frac{\Lambda\left(\frac{1}{2}+it\right)}{|\gamma(s)|}$ (this normalization cancels the exponential decay of $\gamma(s)$ but does not change the set of zeroes). It is real-valued by the functional equation. Making the working hypothesis (to be verified at run-time) that all zeroes are simple, they are found by calculating $Z(t)$ to moderate precision and tracking its sign changes – every sign change implies a zero.

The problem is to prove rigorously that we have found *all* the zeroes. For this let $M(t)$ denote the number of zeroes found up to height t. We know that $M(t) \leq N(t)$, and we would like to assure ourselves that it is not the case that $M(t) \leq N(t) - 1$. For this note that $S(t)$ fluctuates around zero. Turing had observed that if we have "missed a zero" then $\tilde{S}(t) = M(t) - \left(\frac{\theta(t)}{\pi} + 1\right)$ would fluctuate around -1 instead. The fluctuations are easy to smooth:

Theorem 11. *(Littlewood)* $\lim_{T\to\infty} \frac{1}{T}\int_0^T S(t)dt = 0$.

(Turing) Let $h > 0$, $T > 168\pi$. Then $\left|\int_T^{T+h} S(t)dt\right| \le 2.3 + 0.128 \log \frac{T+h}{2\pi}$.

Corollary 12. *If $\frac{2.3}{h} + 0.128\frac{1}{h}\log\frac{T+h}{2\pi} < \frac{1}{2}$ (enough to take $h = c\log T$ for some modest $c > 0$) then the integer nearest $\left|\frac{1}{h}\int_T^{T+h}\tilde{S}(t)dt\right|$ is precisely the number of zeroes missed. In particular, in order to provably find all zeroes up to height T it suffices to evaluate $Z(t)$ up to height $T + c\log T$.*

10.2. **General L-functions.**

Theorem 13. *(Booker) For any L-function (notion defined by a usual set of axioms),*

$$\left|\pi\int_{T_1}^{T_2} S(t)dt\right| \le \frac{1}{4}\log\left|Q\left(\frac{3}{2}+iT_2\right)\right| + \left(\log 2 - \frac{1}{2}\right)\frac{1}{4}\log\left|Q\left(\frac{3}{2}+iT_1\right)\right| + c_\theta d + \frac{d}{\sqrt{2}(X-\sqrt{5})}\,.$$

Here, d is the degree and c_θ is a constant depending on the bound toward the GRC.

10.3. **Verifying RH for $\zeta_K(s)$.** Let $K/\mathbb{Q}$ be a Galois extension with Galois group G. Then

$$\zeta_K(s) = \prod_{\rho\in\hat{G}} \left(L(s;\rho;K/\mathbb{Q})\right)^{\dim\rho}\,.$$

In particular, we can no longer expect the zeroes of $\zeta_K(s)$ to be *simple* and the method above would fail – zeroes are no longer detectable by sign changes. Instead, one needs to verify the GRH for each Artin L-function separately (these are expected to have simple zeroes, a hypothesis that can be checked at run-time). Unfortunately they are not known to be entire, so some care must be taken.

Theorem 14. *(Booker 2005) Up to group-theoretic hypothesis on G there exists an algorithm for checking GRH for $\zeta_K(s)$ up to height T.*

10.4. **Selberg zeta-functions.** The same idea can be used in calculations of Laplace eigenfunctions on a manifold to prove that all eigenfunctions have been found. One simple takes for $S(t)$ the oscillatory error term in Weyl's law and proves a mean-value theorem for it. For a closer analogy think of the Laplace eigenvalues as the zeroes of the Selberg zeta function.

Theorem 15. *(Booker-Strömbergsson, 2008) Let $N(t)$ be the number of Maass waveforms with spectral parameter $r \le t$ on $\Gamma(1)\backslash\mathbb{H}$. Let*

$$S(t) = N(t) - \left(\frac{t^2}{12} - \frac{2t}{\pi}\log\frac{t}{e\sqrt{\frac{\pi}{2}}} - \frac{131}{144}\right),$$

$$E(T) = \left(1 + \frac{c}{\log T}\right)\left(\frac{\pi}{12\log T}\right)^2.$$

Then

$$-2E(T) \le \frac{1}{T}\int_0^T S(t)dt \le E(T)\,.$$

Part 2. Computing automorphic forms on GL(3)

(work of student Ce Bian)

Part 3. GL(n)**?**