MAASS FORMS AND L-FUNCTIONS

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Goal: compute $L$-functions associated to automorphic forms on $GL_2/F$ for $F$ totally real.

Part 1. Fun with the FFT

Using the FFT one can efficiently compute many $L$-values using algorithms for computing a single value.

1. $L(\frac{1}{2} + it)$ for many values of $t$ ($L$ an $L$-function)
2. $L(s; \chi)$ for many values of $\chi$ (current work of student D. Platt)

Using this can test the RH.

1. Intro

Recall the Riemann-Siegel formula ("approximate FE"): let $Z(t)$ be the Riemann zetafunction rotated and rescaled to be real on the real axis. Then:

$$Z(t) = 2\Re \sum_{n \leq \sqrt{t/2\pi}} n^{\frac{1}{2}} e^{(\theta(t) - t \log n)} + O(t^{-1/4}),$$

$$\theta(t) = \frac{t}{2} \log \frac{t}{2\pi e} - \frac{\pi}{8} + O(t^{-1}).$$

Remark 1. In practice, need to refine the formula, taking more terms and getting better control on the error term.

Problem 2. Evaluate this for (many) values of $t$ in an interval $[t_0 - h, t_0 + h]$.

- If $h = O(t_0^{1/2})$, $\sqrt{t/2\pi}$ differs from $\sqrt{t_0/2\pi}$ by a bounded amount, so at the cost of increasing the error term by a constant can have the number of terms fixed.
- Writing $t = t_0 + \delta$ and taking a Taylor expansion in $\theta(t) = \frac{t}{2} \log \frac{t_0}{2\pi e} - \frac{\pi}{8} + O(t_0^{-1}) + \frac{t}{2} \log \frac{t + \delta}{t_0} e$ we have:

$$\theta(t) \approx \frac{t}{2} \log \frac{t_0}{2\pi} - \frac{\pi}{8} - \frac{t}{2} + O \left( \frac{t_0^2 + 1}{t_0} \right).$$
Thus if \( h = O(t_0^{1/4}) \) we have:

\[
Z(t) = 2\Re \left[ e^{i\left(\frac{t}{2} - \frac{1}{2}\right)} \sum_{n \leq \sqrt{t_0/2\pi}} n^{-\frac{1}{2}} e^{it\log \sqrt{\frac{t_0}{2\pi}n}} \right] + O\left(t_0^{-0.1}\right).
\]

Now we have \( Z(t) \) as (roughly) a “band-limited” function – its Fourier Transform has bounded support: the frequencies range between zero (when \( n \) is largest) and \( \frac{1}{2}\log \frac{t_0}{2\pi} \) (when \( n = 1 \))

- The approach works for general L-functions with \( \frac{t_0}{2\pi} \) replaced, in general, with the analytic conductor.

2. THE FOURIER TRANSFORM

For \( f \in S(\mathbb{R}) \) set

\[
\hat{f}(\nu) = \int_{\mathbb{R}} f(t) e^{-i\nu t} dt.
\]

Then (Fourier inversion)

\[
f(t) = \int_{\mathbb{R}} \hat{f}(\nu) e^{i\nu x} d\nu.
\]

<table>
<thead>
<tr>
<th>Function</th>
<th>Fourier transform</th>
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<tr>
<td>( f(t) )</td>
<td>( f(\nu) )</td>
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<tr>
<td>( t \mapsto f(at) )</td>
<td>( \nu \mapsto \frac{1}{a} f\left(\frac{\nu}{a}\right) )</td>
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<td>( (f * g)(t) )</td>
<td>( \hat{f}(\nu) \hat{g}(\nu) )</td>
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<td>( \delta(t) )</td>
<td>( 1 )</td>
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<td>( c(t) = \sum_{n \in \mathbb{Z}} \delta(t-n) )</td>
<td>( \hat{c}(\nu) = \hat{c}(\nu)^2 )</td>
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<tr>
<td>( \hat{f}(t) = \text{sinc}(\pi t) = \frac{\sin(\pi t)}{\pi t} )</td>
<td>( \hat{f}(\nu) = \chi_{[-\frac{1}{2},\frac{1}{2}]}(\nu) )</td>
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Finally, if \( \hat{f}(\nu) \) is supported in \((-1,1)\), then \( \hat{f}(\nu) = \left( \sum_{n \in \mathbb{Z}} \hat{f}(\nu - n) \right) \chi_{[-1/2,1/2]}(\nu) \). Fourier inversion and the Poisson Summation Formula give:

\[
f(t) = \sum_{n \in \mathbb{Z}} f(n) \text{sinc} [\pi(t-n)]
\]

if \( f \) has bandwidth at most \( \frac{1}{2} \). More generally, we obtain:

**Proposition 3.** (“Sampling Theorem”) If \( f(t) \) has bandwidth at most \( B \) (that is, \( \hat{f} \) is supported in \([-B,B]\)), then \( f \) can be recovered by sampling at the rate \( 2B \) (the Nyquist frequency), that is at \( \frac{1}{2B} \mathbb{Z} \):

\[
f\left(\frac{t}{2B}\right) = \sum_{n \in \mathbb{Z}} f\left(\frac{n}{2B}\right) \text{sinc} [\pi(t-n)].
\]

Returning to \( \zeta(t) \), should be enough to sample with frequency \( \frac{1}{2\pi} \log \frac{t}{2\pi} \).

3. L-FUNCTIONS

We think of a degree \( d \) Euler product

\[
L(s) = \prod_p L_p(s) = \prod_{p \in \mathbb{P}} \prod_{j=1}^d \left(1 - \alpha_p^{(j)} p^{-s}\right)^{-1}.
\]
We normalize the coefficients so that the functional equation relates $L(s)$ and $L(1-s)$. In this normalization we usually have $|\alpha_{i,j}^{(s)}| \ll p^\vartheta$ for some $\vartheta \leq \frac{1}{2}$ (unitarity). In particular, we have:

**Conjecture 4.** (Generalized Ramanujan Conjecture) $|\alpha_{i,j}^{(s)}| \leq 1$.

**Theorem 5.** (Kim-Sarnak) For Maass forms one may take $\vartheta = \frac{1}{4}$.

**Definition 6.** Let $\Gamma_R(s) = \pi^{-s/2} \Gamma \left( \frac{s}{2} \right)$, $\Gamma_C = 2(2\pi)^{-s/2} \Gamma(s) = \Gamma_R(s) \Gamma_R(s+1)$. Then:

$$
\Gamma_R(s) = \int_{R^+} e^{-\pi t^2} |t|^s \frac{dt}{|t|}.
$$

For our $L$-function we assume that we have a factor at infinity

$$
L_\infty(s) = \prod_{j=1}^d \Gamma_\infty(s + \mu_j).
$$

The GRC here reads $\Re(\mu_j) = 0$; we’ll assume $|\Re(\mu_j)| \leq \theta < \frac{1}{2}$.

Then $\Lambda(s) = L_\infty(s)L(s)$ satisfies a functional equation

$$
\Lambda(s) = wN^{\frac{1}{2} - s} \Lambda(1-s).
$$

The analytic conductor is then $C(s) = N \prod_{j=1}^d \frac{s+\mu_j}{2\pi}$.

**4. The decay in the Fourier transform of the $L$-function**

Instead write $\gamma(s) = w^{-\frac{1}{2}} N^{\frac{s}{2} - (s - 1)/2} L_\infty(s)$ so the functional equation of $\Lambda(s) = \gamma(s)L(s)$ is $\Lambda(s) = \Lambda(1-s)$, in particular $\Lambda(1/2 + it) = \Lambda(1/2 - it)$. In other words, this function is real on the “real axis”.

Since $|\gamma(1/2 + it)| \sim e^{-\frac{1}{2}Q|t|}$, natural to consider $\Lambda \left( \frac{1}{2} + it \right) e^{\frac{1}{2}Q|t|}$, which is of at most polynomial growth. Would also like to restrict the function to the interval $[t_0 - h, t_0 + h]$ with optimal restriction on the Fourier transform side. In fact we consider

$$
f(t) = \Lambda \left( \frac{1}{2} + it \right) e^{-\frac{1}{2}Q|t|} e^{-(t-t_0)^2/2h^2}.
$$

Use Gaussian since it is an entire function. It also has rapid decay in vertical strips, allowing us to shift contours: set

$$
\hat{f}(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(t) e^{-ixt} dt = \frac{1}{2\pi i} \int_{\Re(s)=\frac{1}{2}} \Lambda(s) e^{-(x+\frac{1}{2}Q)\frac{(s-\frac{1}{2})+(s-\frac{1}{2} - it_0)^2}{2h^2}} ds.
$$

Can shift contours to $\Re(s) = \sigma$. Assuming no poles, get:

$$
\hat{f}(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \Lambda(s+it) \exp \left\{ - \left( x + \frac{i\varpi d}{4} \right) \left( \sigma - \frac{1}{2} + it \right) + \left( \sigma - \frac{1}{2} + i(t - t_0) \right)^2 / 2h^2 \right\} dt,
$$

So

$$
\left| \hat{f}(x) \right| \leq \frac{1}{2\pi} \int_{\Re(s)=\frac{1}{2}} N^{\frac{s}{2} - \frac{1}{2}} \prod_j \left| \Gamma_\infty(s + it + \mu_j) \right| \left| L(s + it) \right| e^{\frac{1}{2}Q|t|} e^{-(t-t_0)^2/2h^2} ds.
$$
Using $\log \Gamma_R(s) = \frac{s-1}{2} \log \frac{s-1}{2\pi e} + \frac{1}{2} \log 2 + O\left(\frac{1}{|s|}\right)$, \(\Re \log \Gamma_R(\sigma + it + \mu_j) = \frac{\sigma-\frac{1}{2}}{2\pi} \log \frac{1}{t} - \frac{\sigma-\frac{1}{2}}{2} + O(1/t)$. Thus
\[
\hat{f}(x) \ll \int dt \ |L(\sigma + it)| \exp \left\{ (\sigma - \frac{1}{2}) \left( \log \sqrt{t/2\pi} - x \right) + \frac{(\sigma - \frac{1}{2})^2}{2t^2} \right\}.
\]
Thus obtain exponential decay for \(x\) large. The bound towards Ramanujan assumed before gives $|L(\sigma + it)| \ll (\zeta(\sigma - \theta))^\tau$ uniformly. Note that inside the log can replace \(t\) with \(t_0\), to obtain the analytic conductor \(Q = Q(\frac{1}{2} + it_0)\). Evaluating the integral gives:
\[
\hat{f}(x) \ll h \exp \left\{ (\sigma - \frac{1}{2}) \left( \log \sqrt{Q} - x \right) + \frac{(\sigma - \frac{1}{2})^2}{2h^2} \right\}.
\]
Optimizing the choice of \(\sigma\) gives:
\[
\hat{f}(x) \ll h \exp \left\{ -\frac{h^2}{2} \left( x - \frac{1}{2} \log Q \right) \right\}.
\]
This was all for \(x > 0\) but since \(f\) is real, \(\hat{f}(-x) = \overline{\hat{f}(x)}\), giving us decay in both directions.

5. Evaluating \(\Lambda(\frac{1}{2} + it)\) from scratch in \([0, T]\).

We have seen that \(f\) is band-limited. We can thus reconstruct \(f\) from a small number of samples. We will work directly with
\[
f(t) = \Lambda \left( \frac{1}{2} + it \right) e^{\frac{i\pi d}{4} \eta t}
\]
with \(\eta \in [0, 1)\) so that \(1 - \eta \sim \frac{1}{T}\). Shifting the contour in the Fourier transform,
\[
\hat{f}(x) = \frac{1}{2\pi} \int_R f(t) e^{-\pi xt}dt
\]
\[
= \frac{1}{2\pi i} \int_{\Re(s)=2} \gamma(s) \left( \sum_{n=1}^\infty a_n n^{-s} \right) \exp \left\{ \left( x + \frac{i\pi \eta d}{4} \right) \left( \frac{1}{2} - s \right) \right\} ds.
\]
We are in the region of absolute convergence so we can exchange summation and integration and get:
\[
= \sum_{n=1}^\infty a_n \frac{1}{2\pi i} \int_{\Re(s)=2} \gamma(s) \exp \left\{ \left( x + \frac{i\pi \eta d}{4} \right) \left( \frac{1}{2} - s \right) \right\} n^{-s}ds.
\]
It follows that
\[
\hat{f}(x) = w^{-\frac{d}{2}} \sum_{n=1}^\infty a_n \sqrt{n} G(x + \log \frac{n}{\sqrt{N}})
\]
where
\[
G(u) = \frac{1}{2\pi i} \int_{\Re(s)=2} \exp \left\{ \left( u + \frac{i\pi \eta d}{4} \right) \left( \frac{1}{2} - s \right) \right\} \prod_{j=1}^d \Gamma_R(s + \mu_j) ds.
\]
**Remark 7.** It seems the sum for \(\hat{f}(x)\) has length \(\sqrt{N}\) but evaluating the decay of \(G\) shows that the correct length is \(\sqrt{Q}\) where \(Q\) is the analytic conductor.
6. From \( \hat{f} \) to \( f \): the DFT

Recall the twisted Poisson summation formula:

\[
\sum_{n \in \mathbb{Z}} f(\alpha + nB) = \frac{2\pi}{B} \sum_{k \in \mathbb{Z}} \hat{f} \left( \frac{2\pi k}{B} \right) e \left( \frac{k\alpha}{B} \right).
\]

Here we use the latter definition of \( \hat{f} \), with \( e^{-ikx} \) rather than \( e(kx) \).

Chose \( A \) such that \( q = AB \in \mathbb{Z}_{>0} \) and let \( \alpha = \frac{m}{A} \), \( m \in \mathbb{Z} \). Then

\[
\sum_{n \in \mathbb{Z}} f \left( \frac{m + nq}{A} \right) = \frac{2\pi}{B} \sum_{k \in \mathbb{Z}} \hat{f} \left( \frac{2\pi k}{B} \right) e \left( \frac{km}{q} \right).
\]

Breaking the \( k \)-sum over residue classes modulo \( q \) we see:

\[
\sum_{n \in \mathbb{Z}} f \left( \frac{m + nq}{A} \right) = \frac{2\pi}{B} \sum_{a(q)} \left( \frac{am}{q} \right) \sum_{k \in \mathbb{Z}} \hat{f} \left( \frac{2\pi a}{B} + 2\pi Ak \right).
\]

Writing \( F(m) = \sum_{n \in \mathbb{Z}} f \left( \frac{m + nq}{A} \right) \) for the LHS, \( \tilde{F}(a) = \sum_{k \in \mathbb{Z}} \hat{f} \left( \frac{2\pi a}{B} + 2\pi Ak \right) \) from the RHS, both are functions on \( \mathbb{Z}/q \mathbb{Z} \) and we have

\[
F(m) = \frac{2\pi}{B} \sum_{a(q)} \left( \frac{am}{q} \right) \tilde{F}(a).
\]

There is an algorithm (the FFT) which calculates all values of \( F(m) \) given the values \( \tilde{F}(a) \) with \( O(q \log q) \) floating-point operations. It is especially efficient if \( q \) is a product of small primes.

We now choose \( A, B \) so that the sums defining \( F, \tilde{F} \) push large values of \( n, k \) into the asymptotic regions. We conclude that, in order to evaluate \( f(t) \) at the \( q \) points \( \frac{\pi}{B} \) it suffices to evaluate \( \hat{f}(x) \) on a grid of points with spacing \( \frac{2\pi}{B} \).

7. From \( G(u) \) to \( \hat{f}(x) \): the DFT, again

Recall the formula

\[
\hat{f}(x) = w^{-\frac{i}{2}} \sum_{n=1}^{\infty} \frac{a_n}{\sqrt{n}} G(x + \log \frac{n}{\sqrt{N}}).
\]

We would like to evaluate this for \( x \) belonging to a grid of spacing \( \frac{2\pi}{B} \). For this let \( \{u_m\} \subset \mathbb{R} \) be a grid of spacing \( \frac{\pi}{NB} \) for some integer \( N \), a refinement of the grid where we are evaluating \( \hat{f}(x) \), and assume we have tabulated values for the first \( K \) derivatives of \( G \) at these grid points. We write \( \epsilon = \frac{\pi}{NB} \) for the half-spacing of the grid, \( I_m \) for the interval \([u_m - \epsilon, u_m + \epsilon]\)

Assume now that \( \log \frac{n}{\sqrt{N}} \in [u_m - \epsilon, u_m + \epsilon] \) for some \( u_m \) in the grid, and let \( x \) be one of the points where we would like to evaluate \( \hat{f}(x) \). Then \( x + u_m \) is also a point in the finer grid, and we have:

\[
G(x + \log \frac{n}{\sqrt{N}}) \approx \sum_{k=0}^{K-1} \frac{G^{(k)}(x + u_m)}{k!} \left( \log \frac{n}{\sqrt{N}} - u_m \right)^k
\]
Note that:

relative precision of more than
precision. A good example is the case of
tributions from the poles of the gamma factor. One gets an expression of the form:

\[ \log \frac{n}{\sqrt{N}} = u_m \]

It follows that

\[ \hat{f}(x) = \sum_{k=0}^{K-1} \frac{1}{k!} \sum_m G^{(k)}(x + u_m) \cdot S_m^{(k)} \]

with

\[ S_m^{(k)} = \log \frac{\sqrt{n}}{\sqrt{\pi}} \left( \log \frac{n}{\sqrt{N}} - u_m \right)^k. \]

Note that:

- The \( S_m^{(k)} \) can be calculated efficiently by running over \( n \) and for each \( n \) identifying to \( S_m^{(k)} \) the next contribution belongs to.
- In the next section we discuss evaluating the \( G^{(k)}(u_m) \).
- We have written \( \hat{f} \) as a sum of \( K \) terms, each of which is a convolution over the fine grid \( u_m \) of the two sequences \( S_m^{(k)} \) and \( G^{(k)}(u_m) \). Such convolutions can be efficiently computed by the FFT.

8. Evaluating \( G(u) \)

By Taylor’s Theorem, if \( u \) is \( \epsilon \)-close to \( u_m \),

\[ G(u) = \sum_{k=0}^{K} \frac{G^{(k)}(u_m)}{k!}(u - u_m)^k + O \left( \max_{|u' - u| \leq \epsilon} \frac{|G^{(k)}(u')|}{k!} \right). \]

It is important to note that we can bound the derivatives of \( G(u) \) independently of \( u \):

\[ G^{(k)}(u) = \frac{1}{2\pi i} \int_{\Im(s)=2} \left( \frac{1}{2} - s \right)^k \exp \left\{ \left( u + i\pi t \right) \Gamma \left( \frac{1}{2} + s \right) \right\} ds, \]

and shifting the contour back to \( \Im(s) = \frac{1}{2} \), we have:

\[ |G^{(k)}(u)| \leq \frac{1}{2\pi} \int \left| \frac{\eta^k}{K!} e^{\frac{2\pi it}{K}} \prod_j \Gamma \left( \frac{1}{2} + it + \mu_j \right) \right| dt. \]

This is small as long as \( \eta \) is close enough to 1: say \( \eta = 1 - \frac{1}{2} \).

Finally, evaluate \( G(u) \) by shifting the contour far to the left, picking up contributions from the poles of the gamma factor. One gets an expression of the form:

\[ G(u) = \sum_{\text{poles } \rho} P_{\rho}(u)e^{\frac{1}{2} - \rho} u. \]

Remark 8. It is important to note that this method entails an enormous loss of precision. A good example is the case of \( e^{-t} = \frac{1}{2\pi} \int_{\Re(s)=2} \Gamma(s)e^{-t} ds \). Shifting the contour all the way to \( \Re(s) = -\infty \) gives the usual power series \( e^{-t} = \sum_{k=0}^{\infty} (-1)^k t^k \).

Here the largest summand is of size about \( e^t \), so to calculate \( e^t \) this way requires relative precision of more than \( 1/e^t \).
Nevertheless in many arithmetic families of $L$-functions the $\Gamma$-factors are basically same for the whole family (say, there are at most finitely many different cases$^2$). It then makes sense to tabulate values of $G(u)$ and its derivatives once and for all, even if the evaluations are expensive. As we shall see below it usually suffices to evaluate $G$ on a fine enough grid, but even if off-grid values are desired they can be found by Taylor expansion.

9. A discrete family

The situation is much simpler when evaluating a family of $L$-functions at a fixed value $s \in \mathbb{C}$. As an example, we shall evaluate the numbers $L(s; \chi)$ as $\chi$ varies over the Dirichlet characters mod $q$, where $q$ is a large integer.

9.1. Zeroth approach. Write the approximate functional equation

$$L(s; \chi) = \sum_{n=1}^{\infty} \chi(n)f_1(n, s) + w_\chi \sum_{n=1}^{\infty} \overline{\chi(n)}f_2(n, 1-s),$$

where both functions $f_1$ and $f_2$ decay for $n \gg \sqrt{q}$ (since $s$ is fixed the analytic conductor is roughly $q$). This is the optimal division for evaluating a single $L(s; \chi)$ – we’d need time about $O(\sqrt{q})$ for each of them, so time $O(q^{3/2})$ for all of them.

9.2. First approach. For $1 \leq a \leq q$ write:

$$\tilde{f}_i(a, s) = \sum_{n=1}^{\infty} f_i(a + nq, s).$$

This sum is in fact very short and can thus be evaluated efficiently: since $f_i$ decays over the scale $\sqrt{q}$, taking just the first term usually suffices.

Now, since $\chi$ is periodic, we have:

$$L(s, \chi) = \sum_{a=1}^{q} \chi(a)f_1(a; s) + \sum_{a=1}^{q} \overline{\chi(a)}f_2(a; 1-s).$$

Furthermore, each of the two sums in the RHS is a DFT in the group $(\mathbb{Z}/q\mathbb{Z})^\times$, and hence can be calculated efficiently in time $O(q \log q)$.

Remark 9. This requires constructing an explicit isomorphism between $(\mathbb{Z}/q\mathbb{Z})^\times$ and a direct product of cyclic groups, but that is easy to construct in time $O(q)$ (for example, if $q$ is prime what is needed is to find a primitive root).

9.3. Second approach. In fact, there is no reason to take the balanced approximate FE – it is better to work with the Dirichlet series directly (recall that it converges in the critical strip). Again breaking the sum into residue classes, we

$^2$Some examples: Dirichlet $L$-functions have two cases, even and odd. For holomorphic modular forms the factor only depends on the weight. For Artin $L$-functions of a particular dimension there are finitely many possibilities.
have:

\[ L(s; \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \]

\[ = \sum_{a=1}^{q} \chi(a) \sum_{n=1}^{\infty} \frac{1}{(a + nq)^s} \]

\[ = q^{-s} \sum_{a=1}^{q} \chi(a) Z\left(\frac{a}{q}, s\right) \]

where \( Z(\alpha, s) = \sum_{n=1}^{\infty} (\alpha + n)^{-s} \) is the Hurwitz zeta-function.

We have thus reduced the problem of evaluating \( L(s; \chi) \) to the problem of evaluating \( Z(\alpha, s) \) where \( \alpha \) ranges over \( \frac{1}{q} \) equally spaced points in \([\frac{1}{q}, 1]\). The latter problem is solved using a Taylor expansion: since \( \frac{d}{d\alpha} Z(\alpha, s) = -s Z(\alpha, s+1) \), we have:

\[ Z(\alpha + \delta, s) = \sum_{j=0}^{\infty} \left( s + j - 1 \right) Z(\alpha, s + j) (-\delta)^j. \]

Since \( Z(\alpha, s + j) \) can be calculated directly when \( j \) is large (region of absolute convergence), we can recursively evaluate \( Z(\alpha, s) \) over finer and finer meshes in a fixed interval.

- Booker’s student David Platt has implemented this in interval arithmetic.

Q. (Rubinstein): Won’t these methods (periodize then use FFT) have problems of accumulated round-off error?

A. (Booker): Can handle this by calculating values of \( Z(\alpha, s) \) in the table to high precision so that we know they are correct. The DFT itself is calculated in double precision.

10. Application: verifying ERH

10.1. The Riemann \( \zeta(s) \). Let \( N(T) \) denote the number of zeroes in the critical strip up to height \( T \), \( \theta(t) \) the accumulated phase of the Gamma factor \( \gamma(s) (\Gamma_R(s)) \) in this case) between \( s = \frac{1}{2} \) and \( s = \frac{1}{2} + it \).

**Fact 10.** \( N(t) \approx \frac{\theta(t)}{\pi} + 1 \), and \( \theta(t) = \frac{1}{2\pi} \log \frac{1}{2\pi r} + \frac{7}{8} + o(1) \).

It is natural to work with \( S(t) = N(t) - \left( \frac{\theta(t)}{\pi} + 1 \right) \).

The zeroes of \( \zeta(s) \) are found by considering the function \( Z(t) = \frac{\Lambda(\frac{1}{2} + it)}{\gamma(s)} \) (this normalization cancels the exponential decay of \( \gamma(s) \) but does not change the set of zeroes). It is real-valued by the functional equation. Making the working hypothesis (to be verified at run-time) that all zeroes are simple, they are found by calculating \( Z(t) \) to moderate precision and tracking its sign changes – every sign change implies a zero.

The problem is to prove rigorously that we have found all the zeroes. For this let \( M(t) \) denote the number of zeroes found up to height \( t \). We know that \( M(t) \leq N(t) \), and we would like to assure ourselves that it is not the case that \( M(t) \leq N(t) - 1 \).

For this note that \( S(t) \) fluctuates around zero. Turing had observed that if we have “missed a zero” then \( \tilde{S}(t) = M(t) - \left( \frac{\theta(t)}{\pi} + 1 \right) \) would fluctuate around \(-1\) instead. The fluctuations are easy to smooth:
Theorem 11. (Littlewood) \( \lim_{T \to \infty} \frac{1}{T} \int_0^T S(t) \, dt = 0. \)

(Turing) Let \( h > 0, T > 168 \pi. \) Then \( \left| \int_T^{T+h} S(t) \, dt \right| \leq 2.3 + 0.128 \log \frac{T+h}{T}. \)

Corollary 12. If \( \frac{2.3}{h} + 0.128 \frac{1}{h} \log \frac{T+h}{T} < \frac{1}{2} \) (enough to take \( h = c \log T \) for some modest \( c > 0) \) then the integer nearest \( \left| \frac{1}{h} \int_T^{T+h} S(t) \, dt \right| \) is precisely the number of zeroes missed. In particular, in order to provably find all zeroes up to height \( T \) it suffices to evaluate \( Z(t) \) up to height \( T + c \log T \).

10.2. General \( L \)-functions.

Theorem 13. (Booker) For any \( L \)-function (notion defined by a usual set of axioms),
\[
\left| \pi \int_{T_1}^{T_2} S(t) \, dt \right| \leq \left| 1 \right| \left| \log Q \left( \frac{3}{2} + iT_2 \right) \right| + \left( \log 2 - \frac{1}{2} \right) \frac{1}{4} \log \left| Q \left( \frac{3}{2} + iT_1 \right) \right| + csd + \frac{d}{\sqrt{2} \left( X - \sqrt{5} \right)}. \]

Here, \( d \) is the degree and \( c_\theta \) is a constant depending on the bound toward the GRC.

10.3. Verifying RH for \( \zeta_K(s) \). Let \( K/\mathbb{Q} \) be a Galois extension with Galois group \( G \). Then
\[
\zeta_K(s) = \prod_{\rho \in \hat{G}} (L(s; \rho; K/\mathbb{Q}))^{\dim \rho}. \]

In particular, we can no longer expect the zeroes of \( \zeta_K(s) \) to be simple and the method above would fail – zeroes are no longer detectable by sign changes. Instead, one needs to verify the GRH for each Artin \( L \)-function separately (these are expected to have simple zeroes, a hypothesis that can be checked at run-time). Unfortunately they are not known to be entire, so some care must be taken.

Theorem 14. (Booker 2005) Up to group-theoretic hypothesis on \( G \) there exists an algorithm for checking GRH for \( \zeta_K(s) \) up to height \( T \).

10.4. Selberg zeta-functions. The same idea can be used in calculations of Laplace eigenfunctions on a manifold to prove that all eigenfunctions have been found. One simple takes for \( S(t) \) the oscillatory error term in Weyl’s law and proves a mean-value theorem for it. For a closer analogy think of the Laplace eigenvalues as the zeroes of the Selberg zeta function.

Theorem 15. (Booker-Strömbergsson, 2008) Let \( N(t) \) be the number of Maass waveforms with spectral parameter \( \tau \leq t \) on \( \Gamma(1) \backslash \mathbb{H} \). Let
\[
S(t) = N(t) - \left( \frac{t^2}{12} - \frac{2t}{\pi} \log \frac{t}{e \sqrt{2}} - \frac{131}{144} \right),
\]
\[
E(T) = \left( 1 + \frac{c}{\log T} \right) \left( \frac{\pi}{12 \log T} \right)^2.
\]

Then
\[
-2E(T) \leq \frac{1}{T} \int_0^T S(t) \, dt \leq E(T).
\]

Part 2. Computing automorphic forms on \( GL(3) \)

(work of student Ce Bian)
Part 3. \( \text{GL}(n) \)?