Math 100:V02 – SOLUTIONS TO WORKSHEET 2
LIMITS

1. Asymptotics

(1) How does the each expression behave when x is large? small? what is x is large but negative? Sketch a plot

(a) $ax^3 - bx^5$ $(a, b > 0)$
Solution: When x is very large, x^5 dominates x^3 so $ax^3 - bx^5 \sim -ax^5$ (which is negative for x positive, positive for x negative!). When x is very small (close to zero), x^3 dominates (is bigger than x^5 though both are very small) and $ax^3 - bx^5 \sim ax^3$.

(b) $e^x - x^4$
Solution: When $x \to \infty$ is very large, $e^x \gg x^4$so $e^x - x^4 \sim e^x$. Near we have $e^x \sim 1 \gg x^4$, so $e^x - x^4 \sim 1$. Finally when x is large but negative ($x \to -\infty$) we have that e^x decays while x^4 grows, so $e^x \ll x^4$ and $e^x - x^4 \sim -x^4$.

(2) Say each expression in words, and then determine its asymptotics near 0 and near ∞.

(a) $e^{|x-5|^3}$
Solution: This is the exponential, of the cube, of the absolute value, of $x - 5$.
For x close to 0, $x - 5 \sim -5$ so $|x - 5| \sim 5$ so $|x - 5|^3 \sim 125$ so $e^{|x-5|^3} \sim e^{125}$. For x very large $x - 5 \sim x$ and since x is positive $|x - 5| \sim |x| = x$ so $|x - 5|^3 \sim x^3$. $e^{|x-5|^3}$ therefore grows roughly like e^{x^3} (in truth e^{x^3} is actually much bigger than $e^{(x-5)^3}$ – the ratio is on the scale of e^{15x^2}– but our expression captures the gist of the growth pattern).

(b) $\frac{1+x}{1+2x-x^2}$
Solution: This is the ratio of (the sum of 1 and x) and (the sum of 1, $2x$, and $-x^2$).
As $x \to 0$ x, x^2 are negligible next to the 1 so $\frac{1+x}{1+2x-x^2} \sim \frac{1}{1} = 1$. As $x \to \infty$ x dominates 1 so $x + 1 \sim x$ and x^2 dominates $x, 1$ so $1 + 2x - x^2 \sim -x^2$. Thus $\frac{1+x}{1+2x-x^2} \sim \frac{x}{-x^2} = -\frac{1}{x}$ – in other words the whole expression decays roughly like $\frac{1}{x}$.

(c) $\frac{e^x+A\sin x}{e^x-x^2}$
Solution: This is the ratio of (the sum of e^x and the product of A and $\sin x$) and (the difference of e^x and x^2).
For x near 0 we have $e^x \sim e^0 = 1$ and $\sin x \to 0$ (we'll later learn that $\sin x \sim x$ near 0) so $e^x + A\sin x \sim 1$ near 0. Similarly $x^2 \sim 0$ so $e^x - x^2 \sim 1$ and we have $\frac{e^x+A\sin x}{e^x-x^2} \sim \frac{1}{1} = 1$. For large x we have $|\sin x| \le 1$ so $A\sin x$ is much smaller than e^x and $e^x + A\sin x \sim e^x$. Simliarly e^x dominates any polynomial including x^2 and we have $e^x - x^2 \sim e^x$. Thus at infinity $\frac{e^x+A\sin x}{e^x-x^2} \sim \frac{e^x}{e^x} = 1$.

(d) $\frac{Ae^{rt}+Be^{-st}}{t+t^2}$ where $r, s > 0$ and $A, B \neq 0$.
Solution: This is the sum of A times the exponential of r times t and B times the exponential of $-s$ times t, all divided by the sum of t and t^2.
As $t \to 0$ we have $t^2 \ll t$ so $t + t^2 \sim t$. $e^{rt} \sim e^0 \sim e^{-st}$ so

$$\frac{Ae^{rt} + Be^{-st}}{t + t^2} \sim \frac{A + B}{t}.$$

As $t \to \infty$, $t^2 \gg t$ while $e^{rt} \gg e^{-st}$ (growing exponential dominates the decaying one!). Thus

$$\frac{Ae^{rt} + Be^{-st}}{t + t^2} \sim \frac{Ae^{rt}}{t^2}.$$

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Conversely as $t \to -\infty$ we have $e^{-st} \gg e^{rt}$ so

$$\frac{Ae^{rt} + Be^{-st}}{t + t^2} \sim \frac{Be^{-st}}{t^2}.$$

(3) Find the asymptotics of the indicated expression at the given point.

(a) $\frac{x^5+Ax^3+x}{Bx^4-x^2}$ as $x \to 0$.
Solution: As $x \to 0$ we have $\frac{x^5+Ax^3+x}{Bx^4-x^2} \sim \frac{x}{-x^2} \sim -\frac{1}{x}$.

(b) $\frac{x^2+1}{x-4}$ as $x \to 3$.
Solution: This is easy: $f(x) \sim \frac{3^2+1}{3-4} = -10$.

(c) $f(x) = \frac{x^2+1}{x-4}$ as $x \to 4$.
Solution: $f(x) \sim \frac{17}{x-4}$.

(d) $f(x) = x^2 - 1$ as $x \to 1$.
Solution: $x^2 - 1 = (1 + (x-1))^2 - 1 = 1 + 2(x-1) + (x-1)^2 - 1 = 2(x-1) + (x-1)^2 \sim 2(x-1)$ as $x \to 1$.

2. Limits

(4) Either evaluate the limit or explain why it does not exist. Sketching a graph might be helpful.

(a) $\lim_{x\to 5}\left(x^3 - x\right)$
Solution: When the function is defined by expression the limit can be obtained by plugging in. $\lim_{x\to 5}\left(x^3 - x\right) = 125 - 5 = 120$.

(b) $\lim_{x\to 1} f(x)$ where $f(x) = \begin{cases} \sqrt{x} & 0 \le x < 1 \\ 3 & x = 1 \\ 2 - x^2 & x > 1 \end{cases}$.
Solution: $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(2 - x^2) = 2 - 1^2 = 1$ and $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \sqrt{x} = \sqrt{1} = 1$ so

$$\lim_{x\to 1} f(x) = 1.$$

(c) $\lim_{x\to 1} f(x)$ where $f(x) = \begin{cases} \sqrt{x} & 0 \le x < 1 \\ 1 & x = 1 \\ 4 - x^2 & x > 1 \end{cases}$.
Solution: $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(4 - x^2) = 4 - 1^2 = 3$ and $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \sqrt{x} = \sqrt{1} = 1$ so the limit does not exist (but the one-sided limits do).

(5) Let $f(x) = \frac{x-3}{x^2+x-12}$.

(a) (Final 2014) What is $\lim_{x\to 3} f(x)$?
Solution: $f(x) = \frac{x-3}{(x-3)(x+4)} = \frac{1}{x+4}$ so $\lim_{x\to 3} f(x) = \frac{1}{3+4} = \boxed{\frac{1}{7}}$.

(b) What about $\lim_{x\to -4} f(x)$?
Solution: The limit does not exist: if x is very close to -4 then $x + 4$ is very small and $\frac{1}{x+4}$ is very large. That said, when $x > -4$ we have $\frac{1}{x+4} > 0$ and when $x < -4$ we have $\frac{1}{x+4} < 0$ so (in the extended sense)

$$\lim_{x\to -4^+} \frac{1}{x+4} = +\infty$$
$$\lim_{x\to -4^-} \frac{1}{x+4} = -\infty.$$

More on this in the next lecture.

(6) Evaluate

(a) $\lim_{x\to\infty} \frac{e^x + A\sin x}{e^x - x^2}$
Solution: By problem 2(c) this is 1.

(b) $\lim_{x\to 0} \frac{e^x + A\sin x}{e^x - x^2}$
Solution: By problem 2(c) this is 1 also.

(c) $\lim_{x\to -\infty} \frac{e^x + A\sin x}{e^x - x^2}$
Solution: By problem 2(c) the numerator is bounded while the denominator grows like x^2, so the whole expression tends to 0.

(7) Evaluate

(a) $\lim_{x\to 2} \frac{x+1}{4x^2-1}$
Solution: The expression is well-behaved at $x = 2$ so $\lim_{x\to 2} \frac{x+1}{4x^2-1} = \frac{2+1}{4\cdot 2^2-1} = \frac{3}{15} = \frac{1}{5}$.

(b) (Final, 2014) $\lim_{x\to -3^+} \frac{x+2}{x+3}$.
Solution: As $x \to -3$ the numerator is close to -1 and while the denominator goes to 0 so the whole expression blows up: we have $\frac{x+2}{x+3} \sim \frac{-1}{x+3}$. Now when $x > -3$ we have $x + 3 > 0$ so the whole expression is negative and $\lim_{x\to -3^+} \frac{x+2}{x+3} = \lim_{x\to -3^+} -\frac{1}{x+3} = -\infty$.

(c) $\lim_{x\to 1} \frac{e^x(x-1)}{x^2+x-2}$
Solution: $\lim_{x\to 1} \frac{e^x(x-1)}{x^2+x-2} = \lim_{x\to 1} \frac{e^x(x-1)}{(x-1)(x+2)} = \lim_{x\to 1} \frac{e^x}{x+2} = \frac{e^1}{1+2} = \frac{e}{3}$.

(d) $\lim_{x\to -2^-} \frac{e^x(x-1)}{x^2+x-2}$
Solution: As $x \to -2$ we have $\frac{e^x(x-1)}{x^2+x-2} = \frac{e^x(x-1)}{(x-1)(x+2)} = \frac{e^x}{x+2} \sim \frac{e^{-2}}{x+2}$ and the expression blows up (we have a vertical asymptote). If $x < -2$ then $x + 2 < 0$ and thus

$$\lim_{x\to -2^-} \frac{e^x(x-1)}{x^2+x-2} = -\infty\,.$$

(e) $\lim_{x\to 1} \frac{1}{(x-1)^2}$
Solution: The function blows up at both sides, and remains positive on both sides. Therefore

$$\lim_{x\to 1} \frac{1}{(x-1)^2} = \infty\,.$$

(f) $\lim_{x\to 4} \frac{\sin x}{|x-4|}$
Solution: $|x-4| \to 0$ as $x \to 4$ while $\sin x \xrightarrow[x\to 4]{} \sin 4 \neq 0$, so the function blows up there. Since $|x-2|$ is positive and $\sin 4$ is negative ($\pi < 4 < 2\pi$) we have

$$\lim_{x\to 4} \frac{\sin x}{|x-4|} = -\infty\,.$$

(g) $\lim_{x\to \frac{\pi}{2}^+} \tan x$, $\lim_{x\to \frac{\pi}{2}^-} \tan x$.
Solution: We have $\tan x = \frac{\sin x}{\cos x}$. Now for x close to $\frac{\pi}{2}$, $\sin x$ is close to $\sin\frac{\pi}{2} = 1$, so $\sin x$ is positive. On the other hand $\lim_{x\to\frac{\pi}{2}} \cos x = \cos\frac{\pi}{2} = 0$ so $\tan x$ blows up there. Since $\cos x$ is decreasing on $[0, \pi]$ it is positive if $x < \frac{\pi}{2}$ and negative if $x > \frac{\pi}{2}$, so:

$$\lim_{x\to\frac{\pi}{2}^+} \tan x = -\infty$$
$$\lim_{x\to\frac{\pi}{2}^-} \tan x = +\infty$$

3. Limits at infinity

(6) Evaluate

(a) $\lim_{x\to\infty} \frac{x^2+1}{x-3}$
Solution: As $x \to \infty$ we have $\frac{x^2+1}{x-3} \sim \frac{x^2}{x} \sim x$ so $\lim_{x\to\infty} \frac{x^2+1}{x-3} = \infty$.

(b) (Final, 2015) $\lim_{x\to -\infty} \frac{x+1}{x^2+2x-8}$
Solution: As $x \to -\infty$ we have $\frac{x+1}{x^2+2x-8} \sim \frac{x}{x^2} \sim \frac{1}{x}$ so $\lim_{x\to -\infty} \frac{x+1}{x^2+2x-8} = 0$.

(c) (Quiz, 2015) $\lim_{x\to -\infty} \frac{3x}{\sqrt{4x^2+x}-2x}$

Solution: As $x \to -\infty$ since $\sqrt{x^2} = |x| = -x$ we have

$$\frac{3x}{\sqrt{4x^2+x}-2x} \sim \frac{3x}{\sqrt{4x^2}-2x} \sim \frac{3x}{2|x|-2x}$$
$$\sim \frac{3x}{2(-x)-2x} \sim \frac{3x}{-4x} = \boxed{-\frac{3}{4}}.$$

and hence $\lim_{x\to-\infty} \frac{3x}{\sqrt{4x^2+x}-2x} = -\frac{3}{4}$.

Solution: Change variables via $x = -y$ with $y \to \infty$. We are then looking at

$$\frac{-3y}{\sqrt{4y^2-y}+2y} \sim -\frac{3y}{\sqrt{4y^2}+2y} \sim -\frac{3y}{2y+2y}$$
$$\sim -\frac{3y}{4y} \sim \boxed{-\frac{3}{4}}.$$

and hence $\lim_{x\to-\infty} \frac{3x}{\sqrt{4x^2+x}-2x} = -\frac{3}{4}$.