Math 100, Lecture 20, 21/3/2024

Last time: Euler Scheme

Input:
(1) ODE $y' = f(y;x)$
(2) Interval $[a,b]$
(3) initial condition $y(a) = y_0$
(4) number of points n $\Leftrightarrow$ step size $h = \frac{b-a}{n}$

Algorithm:
(1) set $x_i = a + ih$ for $i = 0, 1, 2, \ldots, n$
(e.g. $x_0 = a$, $x_n = b$)
(2) For $i = 0, 1, 2, \ldots, n-1$ in order calculate

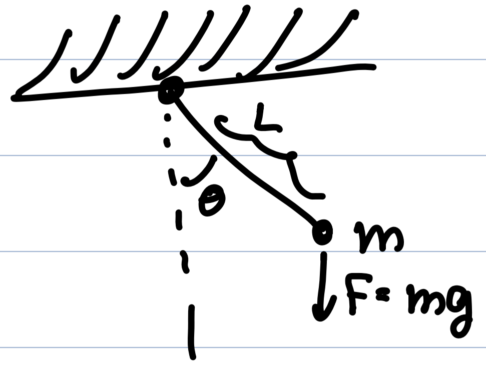
$$\boxed{y_{i+1} = y_i + f(y_i; x_i) \cdot h}$$

Output: points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$, ideally close to the exact solution with initial condition (x_0, y_0)

For reasonable DE (i.e. f), approximation converges to the truth as $h \to 0$ (i.e. as $n \to \infty$)

Implementation:
(1) Spreadsheet
(2) python on ubc.syzygy.ca

Example: Physical pendulum



Physics: torque on rod is

$-L \cdot mg \cdot \sin\theta$

angular velocity $\dot{\theta}$

$\Rightarrow mL^2 \cdot \ddot{\theta} = -Lmg\sin\theta$

$\Rightarrow \boxed{\ddot{\theta} = -\frac{g}{L}\sin\theta}$

If θ is small, can approximate $\sin\theta \approx \theta$

so approximately $\ddot{\theta} = -\omega^2\theta$ $\quad \omega = \sqrt{g/L}$.

Has general solution $\theta(t) = A\sin(\omega t) + B\cos(\omega t)$

If at $t=0$, $\dot{\theta}(0) = 0$, $\theta(0) = \theta_0$, get particular solution

$$\boxed{\theta(t) = \theta_0 \cdot \cos(\omega t)}$$ $\leftarrow$ exact for $\ddot{\theta} = -\omega^2\theta$

Now: compare solutions to $\ddot{\theta} = -\omega^2\theta$

$\ddot{\theta} = -\omega^2\left(\theta - \frac{1}{6}\theta^3\right)$

$\ddot{\theta} = -\omega^2\sin\theta$

Example: $y' = y \cdot (0.3 + 0.1 \cdot \cos(6.28x))$

recall:

$y' = ry$ is the DE for $y(t) = C \cdot e^{rt}$.

Try: $y' = y\left(0.3 + \frac{1}{\sqrt{y}}\right)$ (use numpy.sqrt)

for large y, expect $y' \approx 0.3y$, so $y(x) \approx C \cdot e^{0.3x}$