Math 100, Lecture 18, 14/2/2024

Last time: Optimization

① Closed interval method:

(a) If f is defined, cts on $[a,b]$, f has a global min, global max there. (e.g. f defined by formula)

(b) max & min occur at one of: (1) critical points ($f'(x_0)=0$)
(2) singular pts ($f'(x)$ DNE)
(3) endpoints a, b.

(c) on open/unbounded intervals can argue using asymptotics.

② To solve optimization problem:

(0) read question, draw diagram.

(1) name variables, determine domains = ranges.

(2) find relations between variables, eliminate vars to create objective function.

(3) calculus (see above)

(4) endgame: interpret results, answer question

(4) A standard model for the interaction between two neutral molecules is the *Lennard-Jones Potential* $V(r) = \epsilon\left[\left(\frac{r}{R}\right)^{-12} - 2\left(\frac{r}{R}\right)^{-6}\right]$. Here r is the distance between the molecules and $R, \epsilon > 0$ are parameters.

must have (a) What is the range of r values that makes sense?

$r \geq 0$ since r is a distance, and V is undef at $r=0$ (blowup!) so domain is $(0, \infty)$

(b) Physical systems tend to settle into a state of least energy. Find the minimum of this potential.

$V'(r) = \varepsilon R^{12}(-12)r^{-13} - 2\varepsilon R^6(-6)r^{-7}$

$= 12\varepsilon R^6 r^{-7}\left[-\frac{R^6}{r^6} + 1\right]$

$V'(r) = \varepsilon\left[-12\left(\frac{r}{R}\right)^{-13} + 12\left(\frac{r}{R}\right)^{-7}\right]\frac{1}{R}$

So $V'(r) = 0$ if $1 = \frac{R^6}{r^6}$ if $r = R$.

if $0<r<R$ then $\left(\frac{R}{r}\right)^6 > 1$, $V'(r) < 0$

if $r > R$ then $\left(\frac{R}{r}\right)^6 < 1$, $V'(r) > 0$

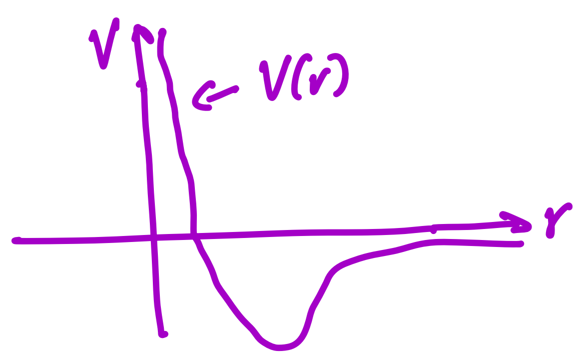
So $r_0 = R$ is the location of the global min

Or: as $r \to 0$, $V(r) \sim \varepsilon\left(\frac{R}{r}\right)^{12} \to \infty$

as $r \to \infty$, $V(r) \sim -2\left(\frac{R}{r}\right)^6 \to 0$

$V(R) = -\varepsilon < 0$

so min at $r = R$



(c) Expand the potential to second order about the minimum.

Expand $V(r) = \varepsilon\left[\left(\frac{r}{R}\right)^{-12} - 2\left(\frac{r}{R}\right)^{-6}\right]$ to 2^{nd} order about R

Know: $V'(r) = \frac{\varepsilon}{R}\left[-12\left(\frac{r}{R}\right)^{-13} + 12\left(\frac{r}{R}\right)^{-7}\right]$

$V''(r) = \frac{\varepsilon}{R^2}\left[156\left(\frac{r}{R}\right)^{-14} - 84\left(\frac{r}{R}\right)^{-8}\right]$

So $V(R) = -\varepsilon$, $V'(R) = 0$, $V''(R) = 72\frac{\varepsilon}{R^2}$

so as $r \to R$, $V(r) \approx -\varepsilon + \frac{1}{2} 72 \frac{\varepsilon}{R^2}(r-R)^2$

$$\approx -\varepsilon + 36\frac{\varepsilon}{R^2}(r-R)^2 =$$

$$\approx -\varepsilon + 36\varepsilon\left(\frac{r}{R}-1\right)^2.$$

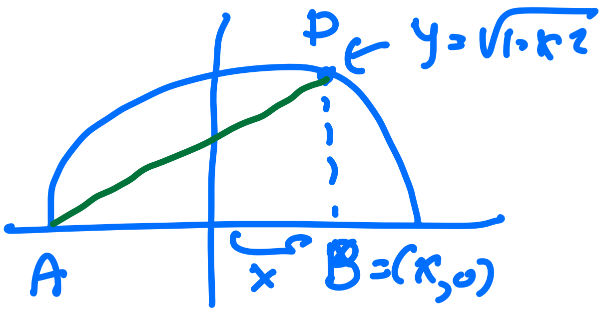
(or: write $r = R + h$

then $\left(\frac{r}{R}\right)^{-6} = \left(\frac{R+h}{R}\right)^{-6} = \left(1+\frac{r}{h}\right)^{-6} = \left(\frac{1}{1+h/R}\right)^6$

$$\approx \left(1 - \frac{h}{R} + \frac{h^2}{R^2}\right)^6 \approx 1 - \frac{6h}{R} + ?\ \frac{h^2}{R^2}$$

$\frac{1}{1-u} \approx 1 + u + u^2 + \ldots$

(6) (Final 2012) The right-angled triangle ΔABP has the vertex $A = (-1, 0)$, a vertex P on the semicircle $y = \sqrt{1 - x^2}$, and another vertex B on the x-axis with the right angle at B. What is the largest possible area of such a triangle?



A = area of triangle then

$$A = \frac{1}{2}(1+x)\sqrt{1-x^2}$$

defined on $[-1, 1]$.

Then $A'(x) = \frac{1}{2}\sqrt{1-x^2} + \frac{1}{2}(1+x)\cdot(1-x^2)^{-\frac{1}{2}}\cdot(-2x)$

$$= \frac{\sqrt{1-x^2}\cdot\sqrt{1-x^2} - 2(1+x)x}{2\sqrt{1-x^2}} = \frac{1-x^2-2x-2x^2}{2\sqrt{1-x^2}}$$

$$= \frac{1-2x-3x^2}{2\sqrt{1-x^2}}$$

zeroes of A' are when $3x^2+2x-1=0$

so $x = \frac{-2 \pm \sqrt{4^2+12}}{6} = -\frac{1}{3} \pm \frac{1}{6}\sqrt{16} = -\frac{1}{3} \pm \frac{4}{6}$

$\in \{\frac{1}{3}, -1\}$

$A(-1) = 0$

$A(+\frac{1}{3}) = \frac{1}{2}\cdot\frac{4}{3}\sqrt{\frac{8}{9}} = \frac{4\cdot 2\cdot\sqrt{2}}{2\cdot 3\cdot 3} = \frac{4\sqrt{2}}{9}$

$A(1) = 0$ so maximum is at $x = \frac{1}{3}$, largest area is $\boxed{4\sqrt{2}/9}$

(7) A ferry operator is trying to optimize profits. Before each ferry trip workers spend some time loading cars after which the trip takes 1 hour. The ferry can carry up to 100 cars, each paying \$50 for the trip. Worker salaries total \$500/hour and the fuel for the trip costs \$250. The workers can load $N(t) = 100\frac{t}{t+1}$ cars in t hours.

(a) How much time should be devoted to loading to maximize profits *per trip*.

Profit if we load for t hours:

$$P(t) = 50 \cdot 100 \cdot \frac{t}{t+1} - 500t - 250 \text{ for } 0 \le t < \infty$$

$$P'(t) = 5000\frac{(t+1)-t}{(t+1)^2} - 500 = 5000\left[\frac{1}{(t+1)^2} - \frac{1}{10}\right]$$

crit pt when $\frac{1}{(t+1)^2} = \frac{1}{10}$ so $t+1 = \sqrt{10}$

so $t = \sqrt{10} - 1$ hours

see: if $t+1 < \sqrt{10}$, $P' > 0$

if $t+1 > \sqrt{10}$, $P' < 0$ so $\sqrt{10}-1$ is global max,

load for $\sqrt{10}-1$ hours

or: $P(0) = -250$, $t \to \infty$ $P(t) \sim -500t$

$P(1) = 2500 - 500 - 250 = 1750 > 0$

so max in interior $\Rightarrow$ at critical pt.