Math 100, Lecture 14, 29/2/2024

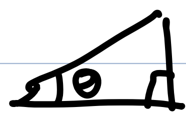
Last time: (1) Log diff: to compute derivative of $y=f(x)$, can hit with log, diff both sides of $\log y = \log f(x)$ wrt x.

(2) Inverse trig: $\theta = \arcsin x$ if $\sin\theta = x$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

def if $|x| \le 1$ $\theta = \arccos x$ if $\cos\theta = x$, $0 \le \theta \le \pi$

def for all x $\rightarrow$ $\theta = \arctan x$ if $\tan\theta = x$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

(*) to compute $\arcsin(\sin\theta)$, $\arccos(\cos\theta)$, $\arctan(\tan\theta)$ use periodicity, symmetry ($\sin(\pi-\theta) = \sin\theta$, $\cos(-\theta) = \cos\theta$) to move θ into correct interval.

(*) to compute $\sin(\arccos x)$ etc (trig (inv trig(x))) create triangle , fill in two sides as $1, x$ compute 3^{rd} side, read off answer.

(*) $\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$; $\frac{d}{dx} \arccos x = -\frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$

Today: Differential equations

Motivation: Most natural laws are differential equations

Goal: (1) what a DE is

(2) what is a solution to a DE

not included: how to solve DE.

modelling (how to create DE)

(3) qualitative study of solutions

Example: $F = ma$ unknown: position $x(t)$

Equation: $F(x) = m \cdot \frac{d^2x}{dt^2}$.

Example: $y' = ry$ solution $y(x) = C e^{rx}$

C constant.

Math 100:V02 – WORKSHEET 12
DIFFERENTIAL EQUATIONS

1. Differential equations

(1) For each equation: Is $y = 3$ a solution? Is $y = 2$ a solution? What are *all* the solutions?

$$y^2 = 4 \qquad ; \qquad y^2 = 3y$$

$3^2 = 9 \neq 4$ ✗

$2^2 = 4$ ✓

all solutions: ± 2

$3^2 = 9 = 3\cdot 3$ ✓

$2^2 = 4 \neq 6 = 3\cdot 2$ ✗

all solutions: $0, 3$

Lesson: to check if y solves equation, plug it in!

(2) For each equation: Is $y(x) = x^2$ a solution? Is $y(x) = e^x$ a solution?

$$\frac{dy}{dx} = y \qquad ; \qquad \left(\frac{dy}{dx}\right)^2 = 4y$$

$\frac{d(x^2)}{dx} = 2x \neq x^2$ ✗

$\frac{d(e^x)}{dx} = e^x$ ✓

$\left(\frac{d(x^2)}{dx}\right)^2 = (2x)^2 = 4\cdot x^2$ ✓

$\left(\frac{d(e^x)}{dx}\right)^2 = (e^x)^2 = e^{2x} \neq 4e^x$ ✗

Date: 29/2/2024, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

lesson: can plug in functions into DE.

intended equality is of functions

(3) Which of the following (if any) is a solution of $\frac{dz}{dt} + t^2 - 1 = z$ (challenge: find more solutions):

A. $z(t) = t^2$; B. $z(t) = t^2 + 2t + 1$

A: $2t + t^2 - 1 \neq t^2$

not a solution

$2t + 2 + t^2 - 1 = t^2 + 2t + 1$

is a solution

(4) Which of the following (if any) is a solution of $\frac{dy}{dx} = \frac{x}{y}$

A. $y = -x$; B. $y = x+5$ C. $y = \sqrt{x^2+5}$

$\frac{d(-x)}{dx} = -1 = \frac{x}{(-x)}$ ✓

$\frac{d(x+5)}{dx} = 1 \neq \frac{x}{x+5}$

not a solution

$\frac{dy}{dx} = \frac{1}{2}\frac{1}{\sqrt{x^2+5}} \cdot 2x$

$= \frac{x}{\sqrt{x^2+5}}$ ✓

Sometimes we know a family of solutions

Can ask for a particular solution from the family.

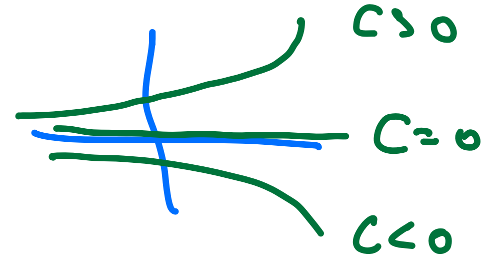
Example: $y = \sqrt{x^2+A}$ all solve $y' = \frac{x}{y}$

which one has $y(1) = 3$?

need $3 = \sqrt{1^2+A}$ so $A=8$. solution is $\sqrt{x^2+8}$

(5) The balance of a bank account satisfies the differential equation $\frac{dy}{dt} = 1.04y$ (this represents interest of 4% compounded continuously). Sketch the solutions to the differential equation. What is the solution for which $y(0) = \$100$?

General solution $y(t) = Ce^{1.04t}$.



Particular solution. $y(0) = Ce^{0} = C$

so need $C = 100$

then $y(t) = 100e^{1.04t}$.

(6) Suppose $\frac{dy}{dx} = ay$, $\frac{dz}{dx} = bz$. Can you find a differential equation satisfied by $w = \frac{y}{z}$? Hint: calculate $\frac{dw}{dx}$.

2. Solutions by massaging and ansatze

(7) For which value of the constant ω is $y(t) = \sin(\omega t)$ a solution of the oscillation equation $\frac{d^2y}{dt^2} + 4y = 0$?

$\dot{y} = \omega \cos(\omega t)$; $\ddot{y} = -\omega^2 \sin(\omega t)$

so want $-\omega^2 \sin(\omega t) + 4\sin(\omega t) = 0$

$\iff (4-\omega^2)\sin(\omega t) = 0$

so if $\omega^2 = 4$ ($\omega = \pm 2$) get zero.

Get solutions $y = \sin(2t)$, $y = \sin(-2t)$.

(actually $A\sin(2t)$ works for all A;
$A = -1$ is like $\omega = -2$
$A = 0$ is like $\omega = 0$)

(General solution: $A\sin(2t) + B\cos(2t)$)