Math 100, lecture 6 , 25/1/2024

Last time:

Saw: Derivative is defined by linear approx

$$\boxed{\begin{array}{ll} f(x+h) \approx f(x) + f'(x)h & \leftarrow \text{linear in } h,\ x \text{ constant} \\ f(x) \approx f(a) + f'(a)(x-a) & \leftarrow \text{linear in } x,\ a \text{ constant} \end{array}}$$

Can also often find f' using formulas, obtain linear approximation

Example: we showed $\boxed{\begin{array}{l} (f+g)' = f'+g' \\ (fg)' = f'g+fg' \end{array}}$ $\boxed{\begin{array}{l} (\alpha f)' = \alpha f' \\ \left(\frac{f}{g}\right)' = \frac{f'g-fg'}{g^2} \end{array}}$

using linear approx

Math 100:V02 – WORKSHEET 6
EXPONENTIAL AND TRIG FUNCTIONS

1. Review: Arithmetic of derivatives

(1) Differentiate

(a) (Final, 2016) $g(x) = x^2 e^x$ (and then also $x^a e^x$)

$g'(x) = 2x \cdot e^x + x^2 e^x$

(b) (Final, 2016) $h(x) = \frac{x^2+3}{2x-1}$

$h'(x) = \frac{2x(2x-1) - (x^2+3)\cdot 2}{(2x-1)^2} = \frac{2x^2 - 2x - 6}{(2x-1)^2}$

$= 2\frac{x^2-x-3}{(2x-1)^2}$ (factor to see where $f'>0$, $f'<0$)

(2) Let $f(x) = \frac{x}{\sqrt{x}+A}$. Given that $f'(4) = \frac{3}{16}$, give a quadratic equation for A.

$f'(x) = \frac{(\sqrt{x}+A) - x\cdot\frac{1}{2\sqrt{x}}}{(\sqrt{x}+A)^2} = \frac{\frac{1}{2}\sqrt{x}+A}{(\sqrt{x}+A)^2}$ ← find f'

interpret $f'(4) = \frac{3}{16}$ as an equation for A

so $f'(4) = \frac{\frac{1}{2}\cdot 2 + A}{(2+A)^2} = \frac{A+1}{(A+2)^2}$ so $\frac{A+1}{(A+2)^2} = \frac{3}{16}$

$3(A^2+4A+4) = 16A+16$ so $3A^2 - 4A - 4 = 0$

so $A = \frac{4 \pm \sqrt{16+48}}{6} = \frac{4\pm 8}{6} = 2, -2/3$ ← solve for A.

Date: 25/1/2024, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(3) Suppose that $f(1) = 1$, $g(1) = 2$, $f'(1) = 3$, $g'(1) = 4$.

(a) What are the linear approximations to f and g at $x = 1$? Use them to find the linear approximation to fg at $x = 1$.

$f(x) \approx f(1) + f'(1)(x-1) = 1 + 3(x-1)$

$g(x) \approx 2 + 4(x-1)$

So $f(x)g(x) \approx (1+3(x-1))(2+4(x-1)) = 1\cdot 2 + (3\cdot 2 + 1\cdot 4)(x-1) + 12(x-1)^2$

$\approx 2 + 10(x-1) + 12(x-1)^2 \approx 2 + 10(x-1)$

correct to 1st order

Or: $(fg)'(1) = f'(1)g(1) + f(1)g'(1) = 3\cdot 2 + 1\cdot 4 = 10$

$(fg)(1) = f(1)g(1) = 1\cdot 2 = 2$

formulas such as product rule, sum rule, .. work point-by-point.

(b) Find $(fg)'(1)$ and $\left(\frac{f}{g}\right)'(1)$.

$\left(\frac{f}{g}\right)' = \dfrac{f'(1)g(1) - f(1)g'(1)}{(g(1))^2} = \dfrac{3\cdot 2 - 1\cdot 4}{2^2} = \dfrac{1}{2}.$

Today: exponentials & trig functions

Facts: $b^{x+y} = b^x b^y$, $(bc)^x = b^x c^x$, $b^0 = 1$

Want to diff b^x at x. Look at $b^{x+h} = b^x \cdot b^h$

note $b^h = b^{0+h} \approx 1 + L(b) \cdot h$ is lin approx at 0,
$L(b) = \frac{d}{dx} b^x \Big|_{x=0}$

so $b^{x+h} \approx b^x (1 + L(b)h) \approx b^x + L(b) b^x \cdot h$
value at x slope

so $\frac{d}{dx}(b^x) = L(b) \cdot b^x$

Fact: $L(b) = \log b$, $\boxed{\frac{d}{dx}(b^x) = (\log b) \cdot b^x}$

e is the number such that $L(e) = 1$ so $\frac{d}{dx} e^x = e^x$

Conclusion: $y = b^x$ has property $y' = r \cdot y$ $r = \log b$

Occurs in population models, disease models, radioactive decay.

write solution to $y' = ry$
as $y(x) = C \cdot e^{rx}$, or $y(x) = C \cdot (e^r)^x$

2. Exponentials

(5) Simplify

(a) $\left(e^5\right)^3$, $\left(2^{1/3}\right)^{12}$, 7^{3-5}.

$= e^{15}$ $\quad = 2^4 = 16$ $\quad = 7^{-2}$

(b) $\log(10e^5)$, $\log(3^7)$. $= 7\log 3$

$= \log(10) + \log(e^5) = \log 10 + 5$

(6) Differentiate:

(a) 10^x

$$\frac{d}{dx}(10^x) = (\log 10)\cdot 10^x$$

(b) $\frac{5\cdot 10^x + x^2}{3^x+1}$ quotient rule

$$\frac{d}{dx}\left(\frac{5\cdot 10^x + x^2}{3^x+1}\right) = \frac{\frac{d}{dx}(5\cdot 10^x + x^2)\cdot(3^x+1) - (5\cdot 10^x + x^2)\frac{d}{dx}(3^x+1)}{(3^x+1)^2}$$

linearity

$$= \frac{\left(5\frac{d}{dx}10^x + \frac{dx^2}{dx}\right)(3^x+1) - (5\cdot 10^x + x^2)\left(\frac{d}{dx}3^x + \frac{d}{dx}\cdot 1\right)}{(3^x+1)^2}$$

$$= \frac{(5(\log 10)\cdot 10^x + 2x)(3^x+1) - (5\cdot 10^x + x^2)((\log 3)\cdot 3^x + 0)}{(3^x+1)^2}$$

exponential rule, power law

3. Trigonometric functions

(7) (Special values) What is $\sin\frac{\pi}{3}$? What is $\cos\frac{5\pi}{2}$?

(8) Derivatives of trig functions

(a) Interpret $\lim_{h\to 0}\frac{\sin h}{h}$ as a derivative and find its value.

(b) Differentiate $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

(9) What is the equation of the line tangent the graph $y = T \sin x + \cos x$ at the point where $x = \frac{\pi}{4}$?