Math 100, Lecture 5

Last time: ① Continuity

(a) Formally, f is cts at a if $\lim_{x \to a} f(x) = f(a)$

(b) informally, "no break" in graph

(c) practically, "most" functions in science (etc) are continuous outside "obvious" points, so

continuity ⟺ gluing function values

② **Derivative**: continuity means (if $f(a) \neq 0$)

(a) that $f(x) \sim f(a)$ as $x \to a$

then $f(x) - f(a) \to 0$ how?

Usually, $f(x) - f(a) \sim C \cdot (x-a)$

$\Leftrightarrow$ $f(x) \approx f(a) + C(x-a)$

$\Leftrightarrow$ $f(a+h) \approx f(a) + ch$

any way call c the derivative (of f at a)

linear approximation

write $f'(a)$, $\left.\frac{df}{dx}\right|_{x=a}$, ..

(b) Call line $y = f(a) + f'(a)(x-a)$ the tangent line (the line tangent to f at $(a, f(a))$).

(c) So: if $f'(a) > 0$, f increasing
$f'(a) < 0$, f decreasing } at a

Math 100:V02 – WORKSHEET 4
CALCULATING DERIVATIVES

1. Definition of the derivative

Definition. $f(a+h) \approx f(a) + f'(a)h$ (or $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$)

(1) Find $f'(a)$ if
  (a) $f(x) = x^2$, $a = 3$.

$f(3+h) = (3+h)^2 = 9 + 6h + h^2 = f(3) + 6h + h^2$

$f(3) = 3^2 = 9$

$\approx f(3) + 6h$ to linear order in h

so $f'(3) = 6$

or $\frac{f(3+h) - f(3)}{h} = \frac{(3+h)^2 - 3^2}{h} = \frac{6h + h^2}{h} = 6 + h \xrightarrow[h\to 0]{} 6$

(using limit definition)

  (b) $f(x) = \frac{1}{x}$, any a.

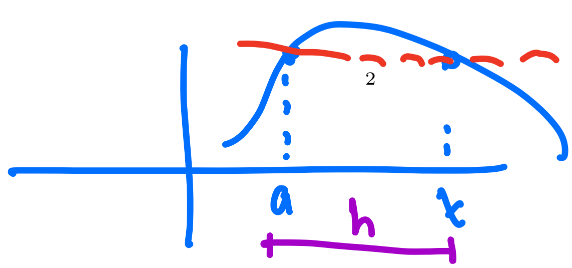
Date: 23/1/2024, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(a) $f(x) = x^3 - 2x$, any a (you may use $(a+h)^3 = a^3 + 3a^2h + 3ah^2 + h^3$).

(2) Express the limits as derivatives: $\lim_{h\to 0} \frac{\cos(5+h) - \cos 5}{h}$, $\lim_{x\to 0} \frac{\sin x}{x}$

$$\frac{d(\cos\theta)}{d\theta}\Big|_{\theta=5}$$

$$\lim_{x\to 0} \frac{\sin x - \sin 0}{x - 0} = \frac{d(\sin\theta)}{d\theta}\Big|_{\theta=0}$$



slope $\frac{f(x) - f(a)}{x - a}$

$x = a + h$, $h = x - a$

Today: compute derivatives

If f is differentiable (= has a derivative) at every point of (a,b) get a derivative function $f'(x)$, df/dx.

Goal: compute using rules.

Facts $\frac{d}{dx}(x^n) = nx^{n-1}$, $\frac{d}{dx}e^x = e^x$

"natural base of the logarithm".

2. The tangent line

(4) (Final, 2015) Find the equation of the line tangent to the function $f(x) = \sqrt{x}$ at $(4, 2)$.

$f(4) = 4^{1/2} = 2$. $f'(x) = \frac{d}{dx}(x^{\frac{1}{2}}) = \frac{1}{2} x^{-\frac{1}{2}}$

So $f'(4) = \frac{1}{2} \cdot 4^{-\frac{1}{2}} = \frac{1}{4}$

So tangent line is $y = \frac{1}{4}(x-4) + 2$

$= 2 + \frac{1}{4}(x-4)$

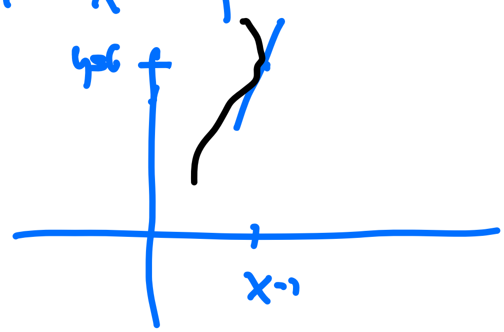
not $f(x)$ $f'(x)$

summary: tangent line has slope $f'(a)$
passes through $(a, f(a))$

$$\boxed{y = f(a) + f'(a)(x-a)}$$

(5) (Final 2015) The line $y = 4x + 2$ is tangent at $x = 1$ to which function: $x^3 + 2x^2 + 3x$, $x^2 + 3x + 2$, $2\sqrt{x+3} + 2$, $x^3 + x^2 - x$, $x^3 + x + 2$, none of the above?

line has slope 4, at $x=1$ passes through $(1, 6)$



(6) Find the lines of slope 3 tangent to the curve $y = x^3 + 4x^2 - 8x + 3$.

(7) The line $y = 5x + B$ is tangent to the curve $y = x^3 + 2x$. What is B?

3. Linear approximation

Definition. $f(a+h) \approx f(a) + f'(a)h$

(8) Estimate

(a) $\star$ $\sqrt{1.2}$

continuity says: $\sqrt{1.2} \approx \sqrt{1} = 1$

Let $f(x) = \sqrt{x} = x^{\frac{1}{2}}$. Know $f(1)$, want $f(1.2)$.

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$. $f'(1) = \frac{1}{2}$ so $f(x) \approx f(1) + \frac{1}{2}(x-1)$ to 1st order

$f(1+h) \approx f(1) + \frac{1}{2}h$

So $f(1.2) \approx 1 + \frac{1}{2}\cdot 0.2 = 1.1$ $\qquad \approx 1 + \frac{1}{2}h$

(b) $\star$ (Final, 2015) $\sqrt{8}$

(c) $\star$ (Final, 2016) $(26)^{1/3}$

Let $f(x) = x^{1/3}$ $\quad f'(x) = \frac{1}{3}x^{-2/3}$

$f(27) = 3$ $\quad f'(27) = \frac{1}{27}$

so $f(26) \approx 3 + \frac{1}{27}\cdot(-1) = 2\frac{26}{27}$

$26-27$, or: $26 = 27 + (-1)$

4. Arithmetic of derivatives

(2) Differentiate

(a) $\star$ $f(x) = 6x^{\pi} + 2x^{e} - x^{7/2}$

(b) $\star$ (Final, 2016) $g(x) = x^2 e^x$ (and then also $x^a e^x$)

Diff rules: Know

f, g functions
a, b constants

(1) Linearity of derivative: $\boxed{(af+bg)' = af' + bg'}$

(2) product rule: $\boxed{(fg)' = fg' + f'g}$

Why are they true?

Say that near x,
$$f(x+h) \approx f(x) + f'(x)h$$
$$g(x+h) \approx g(x) + g'(x)h$$

then

$$(\alpha f + \beta g)(x+h) = \alpha f(x+h) + \beta g(x+h)$$
$$\approx \alpha\big(f(x) + f'(x)h\big) + \beta\big(g(x) + g'(x)h\big)$$
$$\approx \big(\alpha f(x) + \beta f(x)\big) + \big(\alpha f'(x) + \beta g'(x)\big)h$$

$$f(x+h)\,g(x+h) \approx \big(f(x) + f'(x)h\big)\big(g(x) + g'(x)h\big)$$
$$= (fg)(x) + \big(f'(x)g(x) + f(x)g'(x)\big)h +$$
$$+ f'(x)g'(x)h^2$$
$$\approx (fg)(x) + \big((f'g + g'f)(x)\big)\cdot h$$

to 1st order in h

Also $\boxed{\left(\frac{f}{g}\right)' = \frac{f'}{g} - \frac{fg'}{g^2} = \frac{f'g - fg'}{g^2}}$