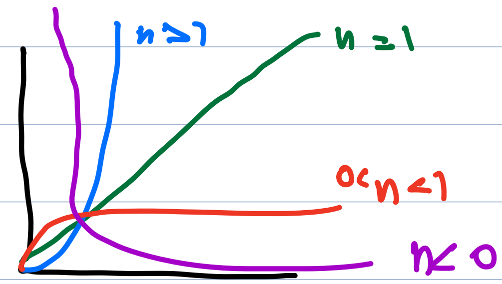
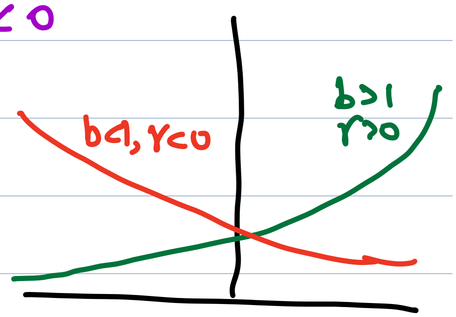
Math 100, Lecture 2

Last time: (1) Power laws Ax^{n} index



(2) Exponentials Ae^{rx} rate, Ab^{x} base



(3) "Ladder of functions"

at ∞, exponentials grow/decay faster than power laws

Today: Asymptotics of expressions

(2) Order the following functions from small to large *asymptotically as $x \to \infty$:*

(a) $1,\ \sqrt{x},\ x^{-1/2}, x^{1/3},\ e^x,\ x^{-1/3},\ 10^6 x^{2024},\ e^{-x},\ e^{x^2},\ \frac{2024}{x^{100}},\ 5^x,\ x.$

$$e^{-x} \ll \frac{2024}{x^{100}} \ll x^{-\frac{1}{2}} \ll x^{-\frac{1}{3}} \ll 1 \ll x^{\frac{1}{3}} \ll x^{\frac{1}{2}} \ll x \ll 10^6 x^{2024} \ll e^x \ll 5^x \ll e^{x^2}$$

towards 0: $10^6 \cdot x^{2024} \ll x \ll x^{\frac{1}{2}} \ll x^{\frac{1}{3}} \ll 1 \sim e^x \sim e^{-x} \sim e^{x^2} \sim 5^x \ll x^{-\frac{1}{3}} \ll x^{-\frac{1}{2}} \ll \frac{2024}{x^{100}}$

(b) Extra: add in $\log x$, $e^{\sqrt{x}}$, $(\log x)^2$, $\log \log x$, $\frac{1}{\log x}$.

log x grows slower than all power laws

(c) Repeat (a), this time as $x \to 0^+$.

Def: Let f, g be functions. Say "f is asymptotic to g in the limit $x \to a$" if $f - g \ll f, g$

$\iff \frac{f}{g} \xrightarrow[x \to a]{} 1.$

Fact: If $f \ll g$ then $f + g \sim g$

Pay attention to signs.

Sketches only as good as the input that goes in

Plot = create using computer.

2. Asymptotics: simple expressions

(3) How does ~~the~~ each expression behave when x is large? small? what is x is large but negative? Sketch a plot

(a) $1 - x^2 + x^4$ ("Mexican hat potential")

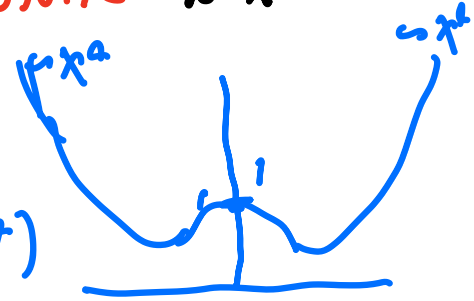
for large x, $1 \ll x^2 \ll x^4$ so $1 - x^2 + x^4 \sim x^4$

Say: as $x \to \infty$, $1 - x^2 + x^4$ is asymptotic to x^4

as $x \to -\infty$, $1 - x^2 + x^4 \sim x^4$

as $x \to 0$, $1 - x^2 + x^4 \sim 1$

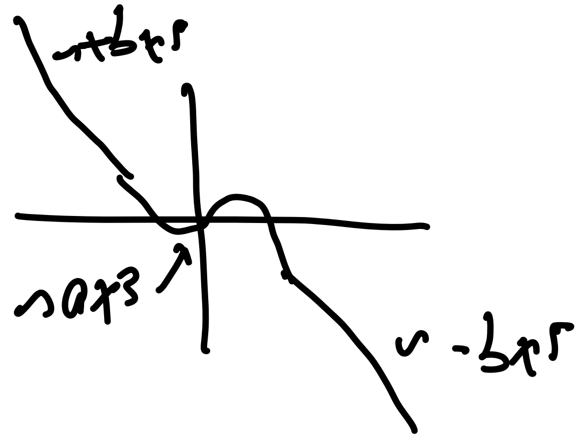
as $x \to 0$, the next correction is $-x^2$ $(x^2 \gg x^4)$



(b) $ax^3 - bx^5$ $(a, b > 0)$

As $x \to \pm\infty$, $x^5 \gg x^3$, so $ax^3 - bx^5 \sim -bx^5$

As $x \to 0$, $ax^3 - bx^5 \sim ax^3$

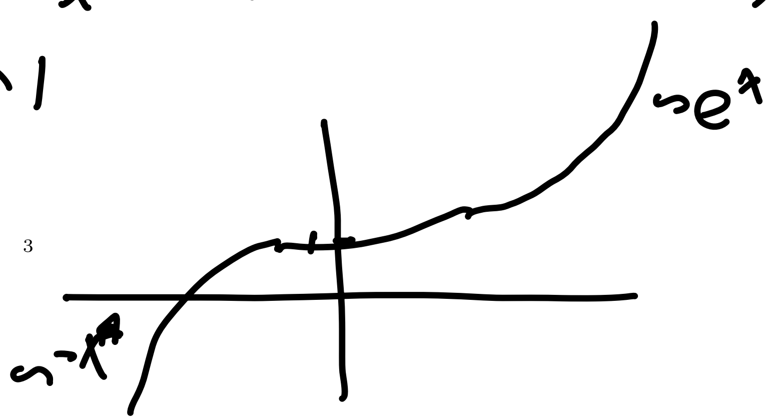


(c) $e^x - x^4$

As $x \to \infty$, $e^x - x^4 \sim e^x$

As $x \to -\infty$ $e^x - x^4 \sim -x^4$ $(e^x \to 0$ as $x \to -\infty)$

As $x \to 0$ $e^x - x^4 \sim 1$



(d) Wages in some country grow at 2% a year (so the wage of a typical worker has the form $A \cdot (1.02)^t$ where t is measured in years and A is the wage today). The cost of healthcare grows at 4% a year (so the healthcare costs of a typical worker have the form $B \cdot (1.04)^t$ where B is the cost today). Suppose that today's workers can afford their healthcare (A is much bigger than B). Will that be always true? Why or why not?

(e) Three strains of a contagion are spreading in a population, spreading at rates 1.05, 1.1, and 0.98 respectively. The total number of cases at time t behaves like

$$A \cdot 1.05^t + B \cdot 1.1^t + C \cdot 0.98^t \, .$$

(A, B, C are constants). Which strain dominates eventually? What would the number of infected people look like?

as $t \to \infty$, $A \cdot 1.05^t + B \cdot 1.1^t + C \cdot 0.98^t \sim B \cdot 1.1^t$

second strain dominates

(4) The (attractive) interaction between two hadrons (say protons) due to the strong nuclear force can be modeled by the *Yukawa potential* $V_{\mathrm{Y}}(r) = -g^2 \frac{e^{-\alpha m r}}{r}$ where r is the separation between the particles, and g, α, m are positive constants. The elecctrical repulsion between two protons is described by the Columb potential $V_{\mathrm{C}}(r) = kq^2 \frac{1}{r}$ where k, q are also positive constants. Which interaction will dominate for large distances? Will the net interaction be attractive or repulsive? Note that g^2 is much larger than kq^2.

Since $e^{-\alpha m r} \ll 1$, as $r \to \infty$ $\quad \frac{e^{-\alpha m r}}{r} \ll \frac{1}{r}$

so $V_C(r) \gg V_Y(r)$ as $r \to \infty$.

As $r \to 0$, $e^{-\alpha m r} \sim 1$ $\quad (e^0 = 1)$

so $\frac{e^{-\alpha m r}}{r} \sim \frac{1}{r}$ as $r \to 0$

so $V_Y(r) \sim -g^2 \frac{1}{r}$, $V_C \sim kq^2 \frac{1}{r}$

so $V_Y(r) + V_C(r) \sim (kq^2 - g^2) \frac{1}{r}$