1A3

9. Optimization (1/11/2023)

Goals.

(1) Review: calculus and the shape of the graph
(2) Optimization of functions
(3) Problem solving: optimization problems

Last Time. **Taylor Expansion**

① To approximate f near/about $x=a$, accurate to nth order

Use $\boxed{T_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots + c_n(x-a)^n}$

$\boxed{c_k = \frac{1}{k!} f^{(k)}(a)}$

as $x \to a$ $\quad f(x) - T_n(x) \ll (x-a)^n$

② Combinations: if T_f, T_g are nth order expansions of f, g about $x=a$, then $\alpha T_f + \beta T_g$, $T_f T_g$ approximate $\alpha f + \beta g$, fg to nth order

③ ~~Building~~ Composition: If $T_g(x)$ approximates $g(x)$ about $x=a$
$T_f(u)$ " $f(u)$ " $u=b$
(both to nth order) $b=g(a)$

Then $T_f(T_g(x))$ approximates $f(g(x))$ to nth order about $x=a$

④ Anchors: $e^u \approx 1 + \frac{u}{1!} + \frac{1}{2!}u^2 + \frac{1}{3!}u^3 + \frac{1}{4!}u^4 + \dots$

$\frac{1}{1-u} \approx 1 + u + u^2 + u^3 + u^4 + \dots$

Plotting and optimization

~~Idea~~ Suppose we have f defined, continuous on $[a,b]$

Fact: f has ~~a~~ global/absolute maximum & minimum values on $[a,b]$

Q: where are these achieved?

A: suppose that $x \in (a,b)$, $f'(x) \neq 0$.
then near x f is either increasing or decreasing,
so $f(x)$ not an extreme value of f.

$\Rightarrow$ Conclusion: max, min only can occur at:

1. critical pts : where $f'(x) = 0$; or
2. singular pts: where $f'(x)$ undef ; or
3. endpoints: where $x = a$ or $x = b$

$\Rightarrow$ Global max value is largest f-value among these pts
(same for min)

Math 100A – WORKSHEET 9
OPTIMIZATION

1. Optimization of functions

(1) Let $f(x) = x^4 - 4x^2 + 4$.

(a) Find the absolute minimum and maximum of f on the interval $[-5, 5]$.

f cts (polynomial), interval is closed

$f'(x) = 4x^3 - 8x = 4x(x^2-2) = 4x(x+\sqrt{2})(x-\sqrt{2})$

$\Rightarrow$ critical pts at $x=0$, $x=\pm\sqrt{2}$

crit. pts $f(0) = 4$

$f(\pm\sqrt{2}) = 0$

end pts $f(\pm 5) = 529$

So max value is 529, attained at ± 5

min " " 0, " " $\pm\sqrt{2}$.

(b) Find the absolute minimum and maximum of f on the interval $[-1, 1]$.

Now only (0,4) is a critical point.

$f(\pm 1) = 1$

So now max is 4, attained at $x=0$

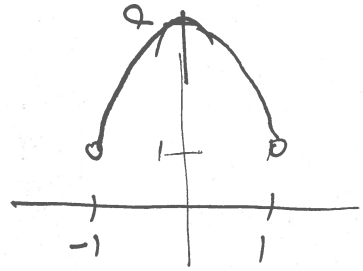
min is 1, " " $x=\pm 1$.

Date: 1/11/2023, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

Point: $f'(x) = 4x(x+\sqrt{2})(x-\sqrt{2})$ still true but $x=0$ only zero in domain.

(c) Find the absolute minimum and maximum of f (if they exist) on the interval $(-1, 1)$.

$f(0) = 4$ is max on $[-1, 1]$ so also on $(-1, 1)$



but no smallest value
(the closer x is to ± 1, the smaller $f(x)$ is)

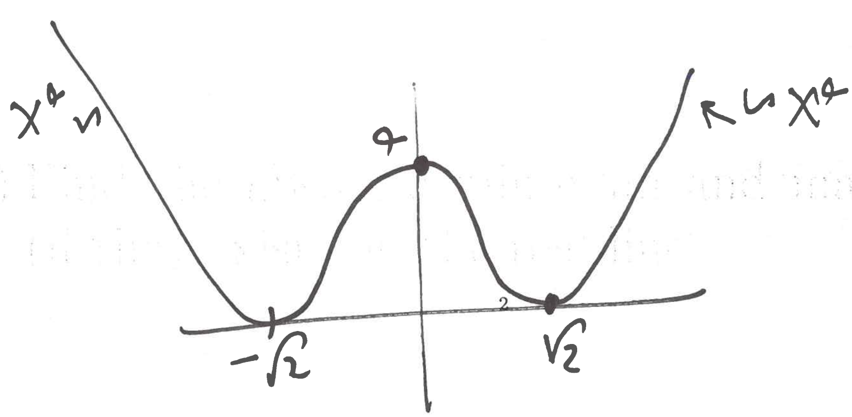
the "infimum" of f on $(-1, 1)$ is 1, but it is not attained

(d) Find the absolute minimum and maximum of f (if they exist) on the real line.

As $x \to \pm\infty$, $f(x) \sim x^4 \to \infty$ so no max.

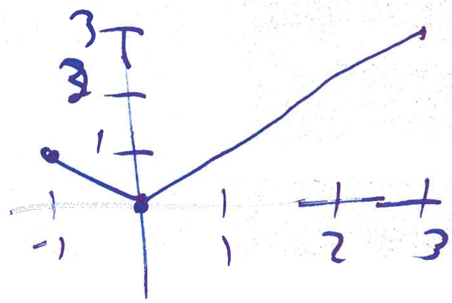
min cannot occur toward ∞ (f is large there)
so in the interior $\Rightarrow$ at critical point.

$\Rightarrow$ min is 0, attained at $\pm\sqrt{2}$.



(2) Let $f(x) = |x|$. Find the absolute minimum and maximum of f on the interval $[-1, 3]$.

(1) don't have to do calculus



(2) don't forget singular points

(3) Find the global extrema (if any) of $f(x) = \frac{1}{x}$ on the intervals $(0, 5)$ and $[1, 4]$.

f is decreasing on $(0, \infty)$

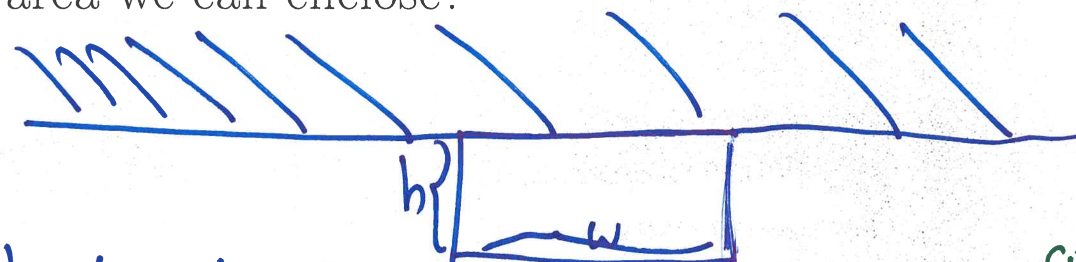
so no extrema on $(0, 5)$,

on $[1, 4]$ max at $x=1$

min at $x=4$

(5) Suppose we have 100m of fencing to enlose a rectangular area against a long, straight wall. What is the largest area we can enclose?

(0) drawing →



(1) nam

Let h, w be the lengths (in meters) of the sides of the rectangle.

Let A be the area of the enclosure in m^2.

At maximal area, $2h + w = 100$m

Also $A = hw = h\underbrace{(100-2h)}_{\text{objective function}}$

relations

And this makes sense if $0 \le h \le 50$ ← domain

$w \ge 0$

include degenerate rectangles at $h=0$, $h=50$

calculus

$A'(h) = 100 - 2h - 2h = 100 - 4h$, crit pt at $h = 25$

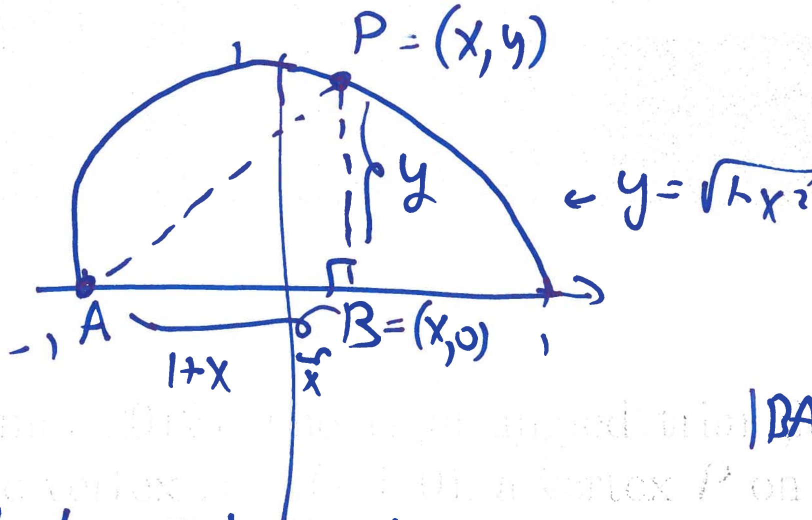
$A(0) = A(50) = 0$, $A(25) = 25 \cdot 50 = 1250\ m^2$

endgame | The largest possible area is $1{,}250\ m^2$.

sanity checks: $h = 25$m, $w = 50$m on scale of 100m,

$A > 0$

(6) ⋆⋆ (Final 2012) The right-angled triangle ΔABP has the vertex $A = (-1, 0)$, a vertex P on the semi-circle $y = \sqrt{1 - x^2}$, and another vertex B on the x-axis with the right angle at B. What is the largest possible area of such a triangle?



$|BA| = x - (-1)$
$= x+1$

S = area of triangle ΔABP

relations: $S = \frac{1}{2}(1+x)\cdot y = \frac{1}{2}(1+x)\sqrt{1-x^2}$

$y = \sqrt{1-x^2}$

domain: $-1 \leq x \leq 1$

$$4S^2 = (1+x)^2(1-x^2) = (1+x)^3(1-x)$$

$d/dx \downarrow$

$$8S'S = 3(1+x)^2(1-x) - (1+x)^3 = (1+x)^2(3-3x-1-x)$$

so $$S' = \frac{(1+x)^2(2-4x)}{4(1+x)\sqrt{1-x^2}} = \frac{(1+x)(1-2x)}{2\sqrt{1-x^2}}$$

So S has critical pt at $x = \frac{1}{2}$

(non-diff at endpoints ± 1)

$S(-1) = S(1) = 0$ (degenerate solutions)

$S\left(\frac{1}{2}\right) = \frac{1}{2} \cdot \frac{3}{2} \cdot \sqrt{\frac{3}{4}} = \frac{3\sqrt{3}}{8}$.

So max of S is $\boxed{\frac{3\sqrt{3}}{8}}$ attained at $x = \frac{1}{2}$.

(sanity check: $y(x) = y(-x)$ but $1-x < 1+x$ if $x > 0$
so always larger area if $x > 0$)

(7) A ferry operator is trying to optimize profits. Before each ferry trip workers spend some time loading cars after which the trip takes 1 hour. The ferry can carry up to 100 cars, each paying \$50 for the trip. Worker salaries total \$500/hour and the fuel for the trip costs \$250. The workers can load $N(t) = 100\frac{t}{t+1}$ cars in t hours.

(a) How much time should be devoted to loading to maximize profits *per trip*.

Revenue: $50 \cdot N(t) = 50 \cdot 100 \cdot \frac{t}{t+1} = 5{,}000 \frac{t}{t+1}$

Costs: $250 + 500t$

Profit: $P(t) = 5{,}000 \frac{t}{t+1} - 500t - 250$ domain $[0, \infty)$

$P(0) = -250$, at $t \to \infty$, $P(t) \sim -500t$

$P(1) = 2{,}500 - 500 - 250 = 1{,}750 > 0$

so max is in middle between $0, \infty$.

$P'(t) = 5{,}000 \frac{1}{(t+1)^2} - 500$ so critical pt if

$(t+1)^2 = 10$, $t = \sqrt{10} - 1$

only critical pt $\Rightarrow$ max at $t = \sqrt{10} - 1$