Math 535, lecture 28, 20/3/2023

Last time: Enumerate $\Delta = \{\alpha_i\}_{i=1}^r$, $r = $ s.s. rk $\mathfrak{g}$.

If $\nu, \nu' \in \mathfrak{t}^*$ say $\nu > \nu'$ if have i s.t. $\nu(\check{\alpha}_i) > \nu'(\check{\alpha}_i)$ and $\nu(\check{\alpha}_j) = \nu'(\check{\alpha}_j)$ if $j < i$.

Saw: If V irrep this gives a total order on weights, positive roots are positive.

(V is a sum of weight spaces $\leftrightarrow$ $\mathfrak{t}$ acts diagonally)

Thm: Let V be a irrep of $\mathfrak{g}$ (f.d.), λ the heighest weight. Then:

(1) $\dim V_\lambda = 1$

(2) $V_\lambda = \{ \underline{v} \in V \mid \mathfrak{n} \cdot \underline{v} = 0 \}$ $\quad \mathfrak{n} = \bigoplus_{\beta > 0} \mathfrak{g}_\beta$ subalg gen by $\{\mathfrak{g}_\alpha\}_{\alpha \in \Delta}$.

(3) Weights of V have form
$$\lambda - \sum_{i=1}^r n_i \alpha_i, \quad n_i \in \mathbb{Z}_{\geq 0}$$

(4) If $w \in W$, $\mu \in \mathfrak{t}^*$, $\dim V_{w\mu} = \dim V_\mu$, all weights satisfy $|\mu| \leq |\lambda|$ equality iff $\mu \in W \cdot \lambda$.

(5) π is determined by λ

<u>Pf</u>: If $X \in \mathfrak{g}_\beta$, $\underline{v} \in V_\mu$, $\pi(X)\cdot\underline{v} \in V_{\mu+\beta}$.
If $\mu = \lambda$, $\beta > 0$, then $\lambda+\beta > \lambda$ so $\lambda+\beta$ isn't a weight of V. $\Rightarrow V_\lambda$ is annihilated by $\mathfrak{n}$.

Choose non-zero $\underline{v}_\lambda \in V_\lambda$. Subrep'n gen by $\underline{v}_\lambda$ is $\mathcal{U}(\mathfrak{g}_\mathbb{C})\cdot\underline{v}_\lambda$. Have $\mathfrak{g}_\mathbb{C} = \bar{\mathfrak{n}} \oplus \mathfrak{t}_\mathbb{C} \oplus \mathfrak{n}$
so (PBW) $\mathcal{U}(\mathfrak{g}_\mathbb{C}) = \mathcal{U}(\bar{\mathfrak{n}})\,\mathcal{U}(\mathfrak{t}_\mathbb{C})\,\mathcal{U}(\mathfrak{n})$

Now $\mathcal{U}(\mathfrak{n})\cdot\underline{v}_\lambda = \mathbb{C}\underline{v}_\lambda$ (all nonconst element kill it)
$\mathcal{U}(\mathfrak{t}_\mathbb{C})\underline{v}_\lambda = \mathbb{C}\underline{v}_\lambda$ (generators act by scalars)

$\Rightarrow \mathcal{U}(\mathfrak{g}_\mathbb{C})\underline{v}_\lambda = \mathcal{U}(\bar{\mathfrak{n}})\cdot\underline{v}_\lambda$.

But (PBW) $\mathcal{U}(\bar{\mathfrak{n}}) = \sum_\mu \mathcal{U}(\bar{\mathfrak{n}})_\mu$ where each weight has form

$$\mu = -\sum_{i=1}^{r} n_i\alpha_i, \quad n_i \in \mathbb{Z}_{\geq 0}$$

(since $\bar{\mathfrak{n}}$ generated by $\{X_{-\alpha}\}_{\alpha\in\Delta}$, so is $\mathcal{U}(\bar{\mathfrak{n}})$)

and only element of weight 0 is 1

But $\mathcal{U}(\bar{\mathfrak{n}})\underline{v}_\lambda = V$ by irred $\Rightarrow$

V contains a unique vector of weight λ (up to scaling)

+ every weight of V has form $\lambda - \sum_{i=1}^{r} n_i \alpha_i$

We know $V_\lambda \subset \{\underline{v} \mid n \cdot \underline{v} = 0\} = \mathcal{U}$

To see that we have equality: since $t_{\mathbb{C}}$ normalizes n, it acts on $\mathcal{U}$. $\Rightarrow$ $\mathcal{U}$ = sum of weight spaces.

Suppose have $v_\mu \in \mathcal{U}$, $\mu < \lambda$.

Then v_μ is annihilated by n, preserved by $t_{\mathbb{C}}$ so

$$\mathcal{U}(t_{\mathbb{C}}) \cdot \mathcal{U}(n) \cdot v_\mu = \mathbb{C} \cdot v_\mu.$$

$$\Rightarrow \quad \mathcal{U}(\mathfrak{g}_{\mathbb{C}}) \cdot v_\mu = \text{sum of weight spaces all of which are } \leq \mu$$

So $\mathcal{U}(\mathfrak{g}_{\mathbb{C}}) v_\mu$ will be a nonzero ($v_\mu \neq 0$) subrep'n of V without v_λ. But V is irred so $\mathcal{U} = V_\lambda$.

Weyl invariance is automatic for rep'n of G. For rep'n of $\mathfrak{g}$ restrict to Lie G_α; classification above showed that weights are inv't by s_α. Claim for W follows since $\{s_\alpha\}_{\alpha \in \Phi}$ generate W.

Let μ be a weight of V, want $|\mu| \le |\lambda|$.
wlog (act by W), μ is dominant $\Leftrightarrow \langle \mu, \alpha_i \rangle \ge 0$ for all i. Also $\lambda = \mu + \sum_{i=1}^{r} n_i \alpha_i$, $n_i \ge 0$.
So:

$$|\lambda|^2 = |\mu|^2 + \left|\sum_{i=1}^{r} n_i \alpha_i\right|^2 + 2\sum_{i=1}^{n} n_i \langle \mu, \alpha_i \rangle$$

$$\ge |\mu|^2$$

, with equality if $\sum_{i=1}^{r} n_i \alpha_i = 0$
i.e. iff all $n_i = 0$, $\mu = \lambda$
in general if $w \cdot \mu = \lambda$ for some $w \in W$.

Remark: In fact the weights of V are in the convex hull of $\{w\lambda\}_{w \in W}$.

That λ determines π can be proved as for $\mathfrak{sl}_2\mathbb{C}$.
(use $\pi(X_\alpha X_{-\alpha} - X_{-\alpha} X_\alpha) = \pi(\check{\alpha})$)

$\Rightarrow$ if we understand $\mathfrak{t}_\mathbb{C} \oplus \mathfrak{n}$ - action on weights μ and above, $\{X_{-\alpha}\}_{\alpha \in S}$-action above μ, if $v_\mu \in V_\mu$ then

$$\pi(X_\alpha)(\pi(X_{-\alpha}) v_\mu) = \pi(\check{\alpha}) v_\mu + \pi(X_{-\alpha}) \pi(X_\alpha) v_\mu$$

Slick alternative proof: Say V, W irreps with highest weight λ, highest weight vectors $\underline{v}_\lambda \in V_\lambda$, $\underline{w}_\lambda \in W_\lambda$.

Then $\underline{v}_\lambda + \underline{w}_\lambda \in (V \oplus W)_\lambda$ is annihilated by $\mathfrak{n}$ so it generates subrep'n $R = \mathcal{U}(\bar{\mathfrak{n}}) \cdot (\underline{v}_\lambda + \underline{w}_\lambda)$ of which it's the unique highest-weight vector

Let $\pi_V : V \oplus W \to V$, $\pi_W : V \oplus W \to W$ be the projections, which are $\mathfrak{g}$-equivariant.

Then $\pi_V(R) \subset V$ is a subrep'n containing $\underline{v}_\lambda$ so all of V.

$\ker \pi_V \restriction_R = R \cap W$ which is a subrep'n of W omitting $\underline{w}_\lambda$, so is $\{0\}$.

$\Rightarrow$ $\pi_V \restriction_R$ is an isom. of R and V.

$\Rightarrow$ $\pi_W \restriction_R$ " " " " " W.

$\Rightarrow$ $V \cong W$.

∎

Example: $\mathfrak{so}_n$ V, W irreps with highest weights λ, μ. Problem: decompose $V \otimes W$ into irreps

(i.e. identify highest weight vectors of irreps)
(i.e. find $v \in V \otimes W$ annihilated by $\mathfrak{n}$)

$$(V \otimes W)_\sigma = \bigoplus_{\rho_1 + \rho_2 = \sigma} V_{\rho_1} \otimes W_{\rho_2}$$

Example: ("Clebsch-Gordan coeff")

Let V_ℓ be the $(2\ell+1)$-dim'l rep'n of $\mathfrak{su}(2)$

Thm: ("addition of angular momentum"):

$$V_\ell \otimes V_{\ell'} \cong \bigoplus_{\substack{|\ell-\ell'| \le \ell'' \le \ell+\ell' \\ \text{same parity}}} V_{\ell''}$$

(coeffs are complex numbers giving the vector in $V_{\ell''}$ of weight $2m$ as combination of

$$\underset{m_1+m_2=m}{Y^{\ell}_{m_1} \otimes Y^{\ell'}_{m_2}}$$ where $Y^\ell_m \in V_\ell$ has weight $2m$