Math 535, lecture 24 10/3/2023

Last time: dual Weyl chamber

$$\mathcal{C} = \{\nu \in \mathfrak{t}^* \mid \nu(\alpha^\vee) > 0 \ \forall \alpha \in \Delta\} = \{\nu \in \mathfrak{t}^* \mid \langle \nu, \alpha \rangle > 0 \ \forall \alpha \in \Delta\}$$

(or $\forall \alpha \in \Phi^+$)

In particular looked at $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha \in \mathcal{C}$:

(1) $\forall \alpha \in \Delta \ \ s_\alpha \rho = \rho - \alpha \Rightarrow$ (2) $\rho(\alpha^\vee) = 1$ (so $\rho \in \mathcal{C}$)

(3) $\forall w \in \Delta \ \ w\rho - \rho \in \mathbb{Z}[\Delta]$ (4) $\rho(\beta^\vee) \in \mathbb{Z}$

(Say ρ is algebraically integral)

$\rho(\Gamma) \subseteq \mathbb{Z}$, $\Gamma = \mathbb{Z}[\{\alpha^\vee\}_{\alpha \in \Delta}] \subset \Lambda$

$\rho \in \Gamma^* \supset \Lambda^*$ but ρ does not have to be a weight.

Today: representation theory.

Part 4: Representation theory of compact lie groups

Setup: G cpt ctd lie gp
$T \subset G$ max'l torus
$\Lambda \subset \mathfrak{t} = \mathrm{Lie}\, T$ integral lattice
$\Lambda^* \subset \mathfrak{t}^\vee$ the weight lattice
$\Phi = \Phi(G : T) \subset \Lambda^*$ the real roots
$\Delta = \{\alpha_i\}_{i=1}^r \subset \Phi$ a system of simple roots
$\Rightarrow$ positive roots Φ^+; Weyl chamber C

$\{\beta^\vee\}_{\beta \in \Phi} \subset \Lambda \subset \mathfrak{t}$ the coroots
$\{w_i\}_{i=1}^r \subset (\mathfrak{t}/\mathfrak{z})^*$ the fundamental weights

$$w_i(\alpha_j^\vee) = \delta_{ij}.$$

$\mathcal{C} \subset \mathfrak{t}^*$ dual chamber
$\rho = \frac{1}{2} \sum_{\beta \in \Phi^+} \beta$ the half sum of positive roots.

implicit: $\mathfrak{g}_{\mathbb{C}} = (\mathfrak{g}_0 = \mathfrak{t}_{\mathbb{C}}) \oplus \bigoplus_{\beta \in \Phi} \mathfrak{g}_\beta$

$\mathfrak{u}_\alpha = \ker \alpha \subset \mathfrak{t}$, $G_\alpha = Z_G(\mathfrak{u}_\alpha)$

Goal: Understand f.d. rep'n $(V, \pi) \in \mathrm{Rep}(G; \mathbb{C})$

Differentiation $\pi \in \mathrm{Hom}(G, GL(V))$
gives a lie algebra hom $d\pi : \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(V)$
which extends to a hom

$$d\pi_{\mathbb{C}} : \mathfrak{g}_{\mathbb{C}} \to \mathrm{End}_{\mathbb{C}}(V)$$

<u>Lemma:</u> (G ctd lie gp)
Let $W \subset V$ be a subspace. Then W is G-inv't iff W is $\mathfrak{g}$-inv't.

<u>Pf:</u> Let $\underline{v} \in W$. If W is G-inv't, $\exp(tX)\underline{v} \in W$ for all t, so $d\pi(X)\underline{v} = \frac{d}{dt}\Big|_{t=0} \exp(tX)\underline{v} \in W$.

If W is $\mathfrak{g}$-inv't, $d\pi(X)^k \underline{v} \in W$ for all k, so

$$\pi(\exp(X))\underline{v} = \exp(d\pi(X))\,\underline{v} = \sum_{k=0}^{\infty} \frac{1}{k!}(d\pi(X))^k \underline{v} \in W.$$

so W is inv't by a generating subset of G, hence by G.

<u>Weights</u>

Restricting π, $d\pi$ to T, $\mathfrak{t}$ we get

$$\mathrm{Res}^G_T V = \bigoplus_{\mu \in \Lambda^*} V_\mu$$

where $V_\mu = \{\underline{v} \in V \mid \forall t \in T : \pi(t)\underline{v} = \chi_\mu(t)\underline{v}\}$

$= \{ \underline{v} \in V \mid \forall H \in \mathfrak{t} : d\pi(H)\underline{v} = 2\pi i \mu(H)\underline{v} \}$

(later: enough to assume V is a $\mathfrak{g}$-module)

<u>Lemma:</u> $\mathfrak{g}_\beta \cdot V_\mu \subset V_{\mu+\beta}$

(special case is $\mathrm{ad}_{X_\beta} \cdot \mathfrak{g}_\gamma \subset \mathfrak{g}_{\beta+\gamma}$)

<u>Pf:</u> Let $H \in \mathfrak{t}$, $X_\beta \in \mathfrak{g}_\beta$, $\underline{v} \in V_\mu$

Then $\pi(H)(\pi(X_\beta)\underline{v}) = \pi(X_\beta)\pi(H)\underline{v} + [\pi(H), \pi(X_\beta)]\underline{v}$

$= \pi(X_\beta) \cdot 2\pi i \mu(H)\underline{v} + \pi([H, X_\beta])\underline{v}$

$= 2\pi i \mu(H) \cdot \pi(X_\beta)\underline{v} + 2\pi i \beta(H) \cdot \pi(X_\beta)\underline{v}$

$= 2\pi i (\mu+\beta)(H) \cdot (\pi(X_\beta)\underline{v})$.

$\blacksquare$

<u>Example:</u> $su(2)$ ($G = SU(2)$ or $SO(3)$, or $SL_2(\mathbb{C})$)

$G = SU(2)$, $\mathfrak{g} = su(2)$, $\mathfrak{g}_\mathbb{C} = sl_2\mathbb{C}$.

Let $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ basis of $\mathfrak{sl}_2\mathbb{C}$
with

$$[h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = h$$

So the unique positive root α has $\alpha(h) = 2$ ($\Rightarrow h$ is the coroot!)

Thm: The Lie algebra $\mathfrak{sl}_2\mathbb{C} = \mathfrak{su}(2)_{\mathbb{C}}$ has a unique irred rep'n of every dimen $n \in \mathbb{Z}_{\geq 1}$. Every f.d. rep'n is completely reducible

Pf: (Uniqueness) Let (π, V) be an irrep of dim n
write $n = 2\ell + 1$, $\ell \in \frac{1}{2}\mathbb{Z}_{\geq 0}$.

$\pi(h) \in \mathrm{End}_{\mathbb{C}}(V)$ has at least one eigenvalue
Let $\lambda \in \mathrm{Spec}(\pi(h))$ have maximal real part.
Let $\underline{v} = v_\ell$ be an eigenvector,

$$\pi(h)\underline{v} = \lambda \underline{v}.$$

Then $\pi(e)\underline{v} \in V_{\lambda+2}$ $\leftarrow$ if $\lambda(h) = \lambda$
then $(\lambda+\alpha)(h) = \lambda + 2$

By choice of λ, $\pi(e)\underline{v} = \underline{0}$.
for $m = \ell - j$, $j \in \mathbb{Z}_{\geq 0}$ write

$$v_m = \pi(f)^m v$$

Since (weight of v_{m-1}) is (weight of v_m) -2 know

$$\pi(h)\cdot v_m = (\lambda + 2m - 2\ell)\, v_m$$

these weights are all distinct, so $\{v_m\}$ are linearly indep if $\neq 0$. But $\dim_{\mathbb{C}} V < \infty$, so have smallest ℓ' s.t. $\pi(f)\, v_{\ell'} = 0$ $(\ell' \geq -\ell)$

<u>Key claim:</u> for $-\ell' \leq m \leq \ell$

$$\pi(e)\, v_m = (\ell - m)(\lambda - \ell + 1 + m)\, v_{m+1}$$

<u>Pf:</u> For $m = \ell$, $\pi(e)\, v_\ell = 0$.

suppose claim holds for v_m, then

$$\pi(e)\, v_{m-1} = \pi(e)\cdot\pi(f)\cdot v_m = \pi(f)\pi(e)\, v_m + [\pi(e), \pi(f)]\, v_m$$

$$= \pi(f)\, (\ell - m)(\lambda - \ell + 1 + m)\, v_{m+1} + \pi(h)\, v_m$$

$$= \left((\ell - m)(\lambda - \ell + 1 + m) + (\lambda + 2m - 2\ell)\right) v_m$$

$$= (\ell - (m-1))(\lambda - \ell + 1 + (m-1))\, v_m$$

$\blacksquare$

Cor: $\mathrm{Span}_{\mathbb{C}}\{v_m\}_{m=-\ell'}^{\ell} \subset V$ is an $\mathfrak{sl}_2\mathbb{C}$-inv't subspace of dim $\ell+\ell'+1 \geq 1$.

By irreducibility this is all of V, which has dim $= 2\ell+1$ so $\ell'=\ell$. Also $\pi(h)$ is diagonalizable in V.

Also, $v_{-\ell-1} = 0$ but $v_{-\ell} \neq 0$

so $0 = \pi(e)\, v_{-\ell-1} = (2\ell+1)(\lambda - 2\ell)\, v_{-\ell}$

so $\lambda = 2\ell$.

Conclusion: in basis $\{v_m\}_{m=-\ell}^{\ell}$ have

$$\begin{cases} \pi(h)\, v_m = 2m \cdot v_m \\ \pi(f)\, v_m = v_{m-1} \quad (0 \text{ if } m=-\ell) \\ \pi(e)\, v_m = (\ell-m)(\ell+m+1)\, v_{m+1} \quad (0 \text{ is } m=\ell) \end{cases}$$

Pf: (Existence): check above maps satisfy the commutation relations