Math 535, lecture 23, 8/3/2023

Last time: Geometry of the roots

For each root $\alpha \in \Phi$ have coroot $\alpha^\vee \in \mathfrak{t}$ s.t.

$$s_\alpha(x) = x - \alpha(x)\alpha^\vee \quad ; \quad s_\alpha^*(\nu) = \nu - \nu(\alpha^\vee)\cdot\alpha$$

Necessarily $\alpha(\alpha^\vee) = 2$. Also $\alpha^\vee \in \Lambda$.

Def: $\Gamma = \sum_{\alpha\in\Phi} \mathbb{Z}\alpha^\vee \subset \Lambda$. $\qquad \Downarrow$

$$n_{\alpha\beta} \overset{\text{def}}{=} \beta(\alpha^\vee) \in \mathbb{Z} \ \ \forall \beta \in \Phi.$$

Fixing Weyl chamber C get $\Phi = \Phi^+ \sqcup \Phi^-$ according to sign on C. Call $\alpha \in \Phi^+$ simple if not sum of positive roots, Δ = set of simple roots

Saw: Every positive root is a sum of simple roots, $\Delta \subset \mathfrak{t}^*$ indep, span is $(\mathfrak{t}/\mathfrak{z})^* = \{\nu \in \mathfrak{t}^* \mid \nu|_\mathfrak{z} = 0\}$

Cor: $\#\Delta = \dim \mathfrak{t}/\mathfrak{z}$ = "semisimple rank".

Lemma: $\{u_\alpha \mid \alpha \in \Delta\}$ are walls of C.

Combinatorial record

The Dynkin diagram of G (actually of $\mathfrak{g}_{\mathbb{C}}$) is the graph with vertex set Δ, $n_{\alpha\beta}n_{\beta\alpha}$ edges between α, β, directed from the shorter to the longer root (if $\|\alpha\| = \|\beta\|$ have single edge)

Ex: The Lie algebras $\mathfrak{g}_{\mathbb{C}}/\mathfrak{z}_{\mathbb{C}}$, $\mathfrak{g}/\mathfrak{z}$ can be recovered from the Dynkin diagrams

Classification shows ctd Dynkin diagrams are of types $A_n, B_n, C_n, D_n, G_2, F_4, E_6, E_7, E_8$.

Today: dual root system, dual Weyl chamber.

Recall: $\Phi \subset \mathfrak{t}^*$ is a root system, $\operatorname{span}_{\mathbb{R}} \Phi = (\mathfrak{t}/\mathfrak{z})^*$.

Each coroot is a functional on $\mathfrak{t}^*$, action of s_α^* is reflection in the hyperplane $\{v \mid v(\check{\alpha}) = 0\}$.

Map back using inner prdt see: $\{\check{\alpha}\}_{\alpha \in \Phi} \subset \mathfrak{t}$ is a root system with reflections s_α
$\Rightarrow$ Weyl group is again W.

Set $C^\vee = \{ \nu \in \mathfrak{t}^* \mid \forall \alpha \in \Delta : \nu(\alpha^\vee) > 0 \}$
$= \{ \nu \in \mathfrak{t}^* \mid \forall \beta \in \Phi^+ : \langle \nu, \beta \rangle > 0 \}$

This is a Weyl chamber for $\Phi^\vee$, so translates by W cover $\mathfrak{t}^*$ (up to taking closure).

Write $\mathcal{C} = \{ \nu \in \mathfrak{t}^* \mid \forall \alpha \in \Phi : \langle \nu, \alpha \rangle \geq 0 \}$ for the closed dual chamber.

Call $\nu \in \mathfrak{t}^*$ dominant if $\nu \in \mathcal{C}$.

<u>Def:</u> The fundamental weights are the basis of $(\mathfrak{t}/\mathfrak{z})^*$ dual to the coroots $\{\alpha^\vee\}_{\alpha \in \Delta}$.

In terms of inner pdt $2 \dfrac{\langle w_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \delta_{ij}$

$\Delta = \{\alpha_i\}_{i=1}^r$.

<u>Def:</u> Call $\nu \in \mathfrak{t}^*$ algebraically integral if it is in the $\mathbb{Z}$-module gen. by fund weights (mod $\mathfrak{z}$)

Since coroots are integral, every $\nu \in \Lambda$ takes integral values on $\check{\alpha}_i$ so is algebraically integral

---

The half sum of the positive roots

Lemma: Let $\alpha \in \Delta$, $\beta \in \Phi^+ \setminus \{\alpha\}$. Then $s_\alpha(\beta) \in \Phi^+$.

Proof: Write $\beta = \sum_i n_i \gamma_i$, γ_i simple
since $\beta \neq \alpha$, root system is reduced. Some $\gamma_{i_0} \neq \alpha$
Then

$$s_\alpha(\beta) = \beta - n_{\alpha\beta} \cdot \alpha = \sum_i n_i \gamma_i - n_{\alpha\beta}\alpha$$

has same coeff of γ_{i_0} as β, and $n_{i_0} > 0$
so all coeff are > 0 and $s_\alpha(\beta) \in \Phi^+$

Def: $\rho = \frac{1}{2} \sum_{\beta \in \Phi^+} \beta \in \mathfrak{t}^*$

Lemma: $\alpha \in \Delta$, $\beta \in \Phi$, $w \in W$.
(1) $s_\alpha \rho = \rho - \alpha \Rightarrow$ (2) $\rho(\check{\alpha}) = 1$
(3) $w\rho - \rho \in \mathbb{Z}[\Delta] = \Gamma \Rightarrow \rho(\check{\beta}) \in \mathbb{Z}$.

Pf: (1) $\rho = \frac{1}{2}\alpha + \frac{1}{2}\sum_{\beta \in \Phi^+ \setminus \{\alpha\}} \beta$ apply s_α.

$s_\alpha(\frac{1}{2}\alpha) = -\frac{1}{2}\alpha$, s_α permutes $\Phi^+ \setminus \{\alpha\}$

(3) if claim holds for $w, w' \in W$ then
$w w'\rho = w(w'\rho - \rho) + (w\rho - \rho) \in w \cdot \mathbb{Z}[\Delta] + \mathbb{Z}[\Delta]$
$= \mathbb{Z}[\Delta]$

$\mathbb{Z}[\Delta] = \mathbb{Z}[\Phi]$ so is W-inv.

So $\{w \mid w\rho - \rho \in \mathbb{Z}[\Delta]\}$ is a subgp of W, containing $\{s_\alpha\}_{\alpha \in \Delta}$ by (b), so all of W.

Given β, have $w \in W$ s.t. $w\alpha = \beta$
(u_β is wall of some chamber, is of form $w \cdot C$)

Then $\rho(\beta^\vee) = \rho(w\alpha^\vee) = (w\rho)(\alpha^\vee) = \underbrace{\rho(\alpha^\vee)}_{=1} + \underbrace{(w\rho - \rho)(\alpha^\vee)}_{\in \mathbb{Z}}$.

Cor: $\rho \in C^*$ ($\nu(\alpha^\vee) = 1 > 0$ for all $\alpha \in \Delta$)

Lemma: $\nu \in \Lambda^*$. Then $\nu + \rho \in C^*$ iff $\nu \in C$

Pf: for $\alpha \in \Delta$, $\nu(\alpha^\vee) \in \mathbb{Z}$ so $\nu(\alpha^\vee) \geq 0$ iff $\nu(\alpha^\vee) + 1 > 0$

$(\nu + \rho)(\alpha^\vee)$