Math 535, Lecture 21, 3/3/2023

Last time: α root of G, $\mathfrak{u}_\alpha = \ker(\alpha) \subset \mathfrak{t}$,

$G_\alpha = Z_G(\mathfrak{u}_\alpha)$, $\overline{G_\alpha} = G_\alpha / \ker(\chi_\alpha) \cong SO(3) / SU(2)$

$\Rightarrow$ only roots proportional to α are $\pm\alpha$
("root system is reduced")

$\Rightarrow$ $\dim \mathfrak{g}_\alpha = 1$

$\Rightarrow$ $\exists\, s_\alpha \in N_{G_\alpha}(T) \subset N_G(T)$ s.t. $Ad(s_\alpha)$ fixed $\mathfrak{u}_\alpha$ elementwise, reflects $\mathfrak{t}$ orthogonally wrt $\mathfrak{u}_\alpha$

$\Rightarrow$ wrt dual action $s_\alpha(\alpha) = -\alpha$

Example: $G = SU(3)$, $T = \{\mathrm{diag}(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) \mid \theta_1+\theta_2+\theta_3 = 0\}$

This is a torus: $\cong \{(s_1, s_2, \frac{1}{s_1 s_2}) \mid s_1, s_2 \in \mathbb{T}^1\}$

Claim: T maximal, $W(G:T) \cong S_3$

Pf: restrict std rep'n of G on $\mathbb{C}^3$ to T.

co-ordinate axes are irreps non-isom: given by weights θ_1, θ_2, θ_3.

$\Rightarrow$ Every $w \in N_G(T)$ must permute these subspaces, each $t \in Z_G(T)$ must permute them trivially, i.e. act in each subspace

$\Rightarrow$ $t \in Z_G(T)$ is diagonal so in T

so $T = Z_G(T)$ i.e. a max'l torus

$\Rightarrow$ $N_G(T)$ = signed permutations

$W(G:T) \cong S_3$

If $G = \{g : g^t g = Id, \det g = 1\}$ diff to set $\mathfrak{g} = \{X : X + X^t = 0, \operatorname{tr} X = 0\}$

let $Y \in sl_3 \mathbb{C}$ has unique rep'n $Y = \frac{Y + Y^t}{2} + \frac{Y - Y^t}{2}$

$$= \frac{Y - Y^t}{2} + i \frac{Y + Y^t}{2i} \in \mathfrak{g} \oplus i\mathfrak{g}$$

see $\mathfrak{g}_{\mathbb{C}} \cong sl_3 \mathbb{C}$, $t = \{ i\,\mathrm{diag}(\theta_1, \theta_2, \theta_3) \mid \theta_1 + \theta_2 + \theta_3 = 0\}$

for $i \neq j$ set $E^{ij} \in sl_3\mathbb{C}$ to be usual elem. matrix. If $H = i\,\mathrm{diag}(\theta_1, \theta_2, \theta_3)$ then

$$[H, E^{ij}] = i(\theta_i - \theta_j)E^{ij}$$

so roots are $e_{ij}(H) = \theta_i - \theta_j$

Observe that the Frobenius = Hilbert-Schmidt norm $\|A\|_F = \sum_{ij} |A_{ij}|^2$ is G-inv't.

$\Rightarrow$ o.n.b. of t (divide by i):

$$\frac{1}{\sqrt{6}}(1, 1, -2), \quad \frac{1}{\sqrt{2}}(1, -1, 0)$$

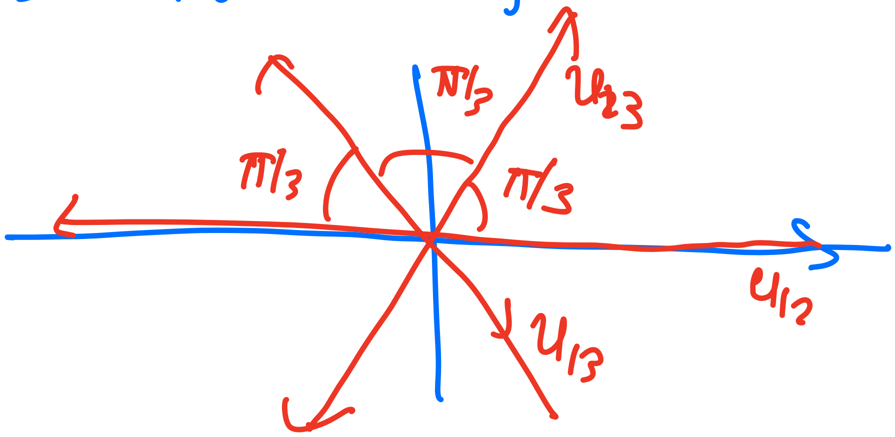
write $H = \frac{x}{\sqrt{6}}(1,1,-2) + \frac{y}{\sqrt{2}}(1,-1,0)$ Now

$e_{12}(H) = \sqrt{2}\, y$, $e_{23}(H) = \sqrt{\frac{3}{2}}\, x - \frac{1}{\sqrt{2}} y$

$$e_{13}(H) = \sqrt{\frac{3}{2}}\, x + \frac{1}{\sqrt{2}} y$$

in co-ords x, y $u_{12} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^\perp$, $e_{13} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}^\perp$

$u_{23} = \begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix}^\perp$, These lines have slope $\pi/3$, $2\pi/3$.



The u_α divide $\mathfrak{t}$ into six cones called Weyl chambers (call u_α the "walls")

<u>note</u>: S_3 acts simply transitively on chambers.

<u>Ex</u>: Do the same for $SU(n)$, $SO(2n)$, $SO(2n+1)$, $Sp(n)$.

<u>Observation</u>: α changes sign on u_α

so chambers $=$ ctd cpnts of $\mathfrak{t} \setminus \bigcup_\alpha u_\alpha$ are exactly sets of constant sign for α

<u>Today</u>: <u>Study Weyl chambers</u>

complement of u_α consists of Half-spaces $h_\alpha^+ = \{H \mid \alpha(H) > 0\}$, $h_\alpha^- = \{H \mid \alpha(H) < 0\}$.

for any fcn $f: \Phi \to \{+, 0, -\}$

define $C_f = \{ H \in \mathfrak{t} \mid \forall \alpha : \operatorname{sgn}(\alpha(H)) = f(\alpha) \}$

then $\mathfrak{t} = \coprod_f C_f$, each C_f = intersection of half-planes & hyperplanes

$\Rightarrow$ convex cone.

in particular $\mathfrak{t} \setminus \bigcup_\alpha \mathcal{U}_\alpha = \coprod_{\substack{f \text{ valued} \\ \text{in } \{\pm\}}} C_f$

those C_f are open cones (intersections of half spaces).

Call those the (open) Weyl chambers in $\mathfrak{t}$.

Call $\mathcal{U}_\alpha$ a wall of a chamber C if

$\dim(\mathcal{U}_\alpha \cap \overline{C}) = \dim \mathcal{U}_\alpha - 1 = \operatorname{rk} G - 2$.

More generally a co-dim k facet of $C = C_f$ is any of the C_g where $C_g \subset \overline{C_f}$, $\dim C_g = \dim C_f - k$

every facet has form $(\mathcal{U}_{\alpha_1} \cap \mathcal{U}_{\alpha_2} \cap \cdots \cap \mathcal{U}_{\alpha_k} \cap \overline{C})^\circ$ (interior as a subset of $\mathcal{U}_{\alpha_1} \cap \cdots \cap \mathcal{U}_{\alpha_k}$.

Then $\overline{C}$ = disjoint union of all facets of C.

Aside: these are Weyl chambers in $\mathfrak{t}$,
roots lie in $\mathfrak{t}^*$

Fix a chamber C. Set $\Delta = \{\alpha \mid \begin{smallmatrix}\alpha|_C > 0 \\ \mathcal{U}_\alpha \text{ is a wall of } C\end{smallmatrix}\}$
(note: $\mathcal{U}_\alpha = \mathcal{U}_{-\alpha}$ exactly one of $\pm\alpha$ is positive in C)

Facts $C = \{H \mid \forall \alpha \in \Delta : \alpha(H) > 0\}$ (later)

Call Δ a system of simple roots.

Observe: $W = W(G;T)$ acts on G by autos, preserving T. Must permute roots, their kernels, hence Weyl chambers

Lemma: The group $W' = \langle \{s_\alpha\}_{\alpha \in \Delta} \rangle$ acts transitively on the Weyl chambers

Pf: Fix $x \in C$, let C' be any chamber, let $y \in C'$. Chambers are either equal or disjoint (equivalence classes for W-invt equiv. relation)
so either $wC' = C$ for some $w \in W'$
or $wy \notin C$ for all $w \in W'$.

Since W' finite have $w \in W'$ with $\|wy-x\|$ is minimal (G-inv't norm on $\mathfrak{g}$)

Then have some wall $\mathcal{H}_\alpha$, $\alpha \in \Delta$ s.t. wy, x are on opposite sides of Δ. Decompose x, wy into components parallel & perp to $\mathcal{H}_\alpha$.

so $x = x_{\parallel} + x_{\perp}$, $wy = y_{\parallel} - y_{\perp}$, $x_\perp, y_\perp > 0$

then $\|x - wy\|^2 = \|x_{\parallel} - y_{\parallel}\|^2 + (x_\perp + y_\perp)^2$

apply s_α to wy get $\underset{\substack{\cap \\ W'y}}{s_\alpha wy} = y_{\parallel} + y_\perp$

and $\|x - s_\alpha wy\|^2 = \|x_{\parallel} - y_{\parallel}\|^2 + |x_\perp - y_\perp|^2$
$< \|x - wy\|^2$
$\Rightarrow\Leftarrow$.

<u>Cor</u>: W acts transitively ($W \supset W'$)

<u>Lemma</u>: W acts simply transitively on chambers

<u>Pf</u>: let $S = \mathrm{Stab}_W(C)$ (setwise stabilizer)
S finite group of affine maps $C \to C$

so S fixes a point $x \in C$
(e.g. $x = \frac{1}{\#S} \sum_{s \in S} s \cdot y$, $y \in C$)

Think of x as element of $\mathfrak{t}$. We have
$Ad_w(x) =$ for all $w \in S$, so $S \subset Z_G(x)/Z_G(T)$

<u>But:</u> (1) $\alpha(x) \neq 0$ for all $\alpha \in \Phi$, so $Z_{\mathfrak{g}_{\mathbb{C}}}(x) = \mathfrak{t}_{\mathbb{C}}$
$\Rightarrow Z_{\mathfrak{g}}(x) = \mathfrak{t}$.

(2) $Z_G(x)$ ctd, and $\mathrm{Lie}(Z_G(x)) = Z_{\mathfrak{g}}(x)$
$\Rightarrow Z_G(x) = T$
and $S \subset T/T = \{1\} \subset W$.

$\blacksquare$

<u>Cor:</u> (Thm: equality of alg. & anal. Weyl gps)

$W' = W$.

<u>Pf:</u> Let $w \in W$. By transitivity of W' have
$w' \in W'$ st $wC = w'C \Rightarrow w^{-1}w'C = C$

by simple transitivity $w^{-1}w' = id$, $w = w'$

Cor: (HW) For any $H \in t$,

$$\mathrm{Stab}_W(H) = \langle \{ s_\alpha \mid \alpha(H) = 0 \} \rangle$$