Math 535, lecture 16 13/2/2023

Last time: G cpt ctd Lie gp, $T \subset G$ torus

tools: (1) $\exp: \mathfrak{g} \to G$ surjective

(2) $\mathrm{Hom}(\mathbb{R}^n/\mathbb{Z}^n; \mathbb{R}^m/\mathbb{Z}^m) \simeq \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z}^m)$

(3) T, $T \times C_m$ are topologically gen. by one element.

Lemma: $\mathfrak{t} = \mathrm{Lie}\, T$ then

(1) $Z_G(T)$ ctd
(2) $Z_G(\mathfrak{t}) = Z_G(T)$
(3) $\mathrm{Lie}\, Z_G(T) = Z_{\mathfrak{g}}(\mathfrak{t})$
(4) $N_G(T)^\circ = Z_G(T)$

Pf: (1) If $g \in Z_G(T)$ then $\overline{\langle g, T \rangle}$ is one-generated $\Rightarrow$ contained in torus.

$\Rightarrow$ $Z_G(T) =$ union of tori, hence ctd.

Today: continue.

(2) $g \in Z_G(T)$ then $\mathrm{Ad}_g \in \mathrm{Aut}(\mathfrak{g})$ is trivial
so its derivative $\mathrm{Ad}_g|_{\mathfrak{t}}$ is trivial

Conversely $\mathrm{Ad}_g|_{\mathfrak{t}}$ being trivial means also

$$\mathrm{Ad}_g(\exp X) = \exp(\mathrm{Ad}_g X) = \exp(X)$$

so Ad_g is trivial on a nbd of the identity in T, hence in all of T. (T is ctd)

(argument only used that G, T are Lie grps with T ctd)

(3) If $X \in Z_{\mathfrak{g}}(\mathfrak{t})$ then for any $s \in \mathbb{R}$, $Y \in \mathfrak{t}$

$$\mathrm{Ad}_{\exp(sX)} \cdot Y = \exp(s\,\mathrm{ad}_X) \cdot Y = \exp(\underset{\underset{\mathrm{End}_{\mathrm{Sp}}(\mathfrak{t})}{\uparrow}}{0}) \cdot Y = Y.$$

so $\exp(sX) \in Z_G(\mathfrak{t})$ for all s,
so $X \in \mathrm{Lie}\, Z_G(\mathfrak{t})$.

Conversely, if $X \in \mathrm{Lie}(Z_G(\mathfrak{t}))$ then $\mathrm{Ad}_{\exp(sX)} \in Z_G(\mathfrak{t})$
for all t.

so $\frac{d}{ds} \mathrm{Ad}_{\exp(sX)}|_{\mathfrak{t}} = 0 \iff \mathrm{ad}_X|_{\mathfrak{t}} = 0$
$\Rightarrow X \in Z_{\mathfrak{g}}(\mathfrak{t})$.

(again used nothing)

(4) Let $N_G(T)$ act on T via adjoint map. This is a cts hom $N_G(T) \to \mathrm{Aut}(T) \cong GL_n(\mathbb{Z})$ if $n = \dim T$.

The image of $N_G(T)^\circ$ must lie in $GL_n(\mathbb{Z})^\circ = \{e\}$ ($GL_n(\mathbb{Z})$ is discrete) so $N_G(T)^\circ$ acts via the trivial automorphism, so it lies in $Z_G(T)$.

Conversely $Z_G(T) \subset N_G(T)$ & is connected, so $Z_G(T) \subset N_G(T)^\circ$. ∎

<u>Cor</u>: Let T be a maximal torus. Then $Z_G(T) = T$.

<u>Pf</u>: Saw: if $g \in Z_G(T)$ then g, T jointly contained in a torus. But T maximal, so $g \in T$.

<u>Def</u>: The **Weyl group** of G is the group $W = W(G:T) = N_G(T)/Z_G(T)$ when T is a max'l torus

(A discrete gp: $N_G(T)/N_G(T)^\circ$) ⨍ for any H, $H^\circ \lhd H$, closed)

(finite since $N_G(T)$ closed $\Rightarrow$ cpt, so $W(G;T)$ is cpt + discrete)

<u>Thm</u>: All maximal tori in G are conjugate

<u>Pf</u>: Let S, T be maximal tori, let $X \in \mathrm{Lie}\, S$, $Y \in \mathrm{Lie}\, T$ be generic elements

(exp X, exp Y generate dense subgps, or $\exp tX$, $\exp tY$ are dense subgps)

Equip $\mathfrak{g} = \mathrm{Lie}\, G$ with an invariant inner pdt, consider

$$f(g) = \| \mathrm{Ad}(g) X - Y \|^2$$

This is a smooth fcn on the cpt manifold G, so has a minimum.

Now

$$f(g) = \|Ad(g)X\|^2 + \|Y\|^2 - 2\langle Ad(g)X, Y\rangle$$
$$= \|X\|^2 + \|Y\|^2 - 2\langle Ad(g)X, Y\rangle$$

$\|\cdot\|^2$ is G-inv't.

So want to maximize $\langle Ad(g)X, Y\rangle$.

Diff wrt g: derivative at g_0 in direction Z is

$$0 = \langle ad_Z \cdot Ad(g_0)X, Y\rangle = \langle ad Z \cdot X_0, Y\rangle$$

if $f(g_0)$ is minimal

$X_0 = Ad(g_0)X$.

$$= \langle [Z, X_0], Y\rangle = -\langle [X_0, Z], Y\rangle$$

$$= -\langle ad_{X_0} \cdot Z, Y\rangle = \langle Z, ad_{X_0} \cdot Y\rangle$$

Used that is π is unitary ($\langle \pi(g)v, w\rangle = \langle v, \pi(g^{-1})w\rangle$)

then $d\pi$ is antisymmetric. $\langle d\pi(X)v, w\rangle = -\langle v, d\pi(X)w\rangle$

$\Rightarrow$

at minimum $\langle Z, [X_0, Y]\rangle = 0$
for all $Z \in \mathfrak{g}$.

So $[X_0, Y] = 0$, i.e. $X_0 \in \mathfrak{z}_{\mathfrak{g}}(Y)$

So X_0 commutes with $\exp(tY)$, so with T.
but then $\exp(sX_0)$ commutes with T, so

so torus $\mathrm{Ad}(g_0) S = \overline{\exp(sX_0)} \subseteq Z_G(T) = T$

But S, T are maximal tori, so $\mathrm{Ad}(g)S = T$.

Example: $G = U(n) \subset GL_n(\mathbb{C})$
$T = U(1)^n$ = diagonal matrices

Corollary: map $T/W \to G/\mathrm{Ad}(G)$ is a homeomorphism

conjugation actions

Pf: Clearly well-defined, cts. Surjective by conjugacy of maximal tori.

Conversely let $t, t' \in T$ be conjugate in G
Say $g t' g^{-1} = t$ then $t \in gTg^{-1}$ so T, gTg^{-1}
both maximal tori in G, hence also in $Z_G(t)$,

thus in $Z_G(t)^\circ$. By conjugacy of max'l tori in $Z_G(t)^\circ$, $\exists\, z \in Z_G(t)^\circ$ s.t. $zgTg^{-1}z^{-1} = T$

$\Rightarrow (zg)T(zg)^{-1} = T$ so $zg \in N_G(T)$

also $(zg)t'(zg)^{-1} = z(gt'g^{-1})z^{-1} = ztz^{-1} = t$

$z \in Z_G(t)$

so t, t' conjugate by element in $N_G(T)$ so by class in W.

∎

(implicit: since $N_G(T)^\circ = Z_G(T)$ acts trivially action of $N_G(T)$ on T factors through $W = N_G(T)/Z_G(T)$)

Example: $G = U(n)$, $T = U(1)^n$ (diagonal torus) $W = S_n$ (as the permutation matrices)

map $G/\mathrm{Ad}(G) \to T/W$ is the multiset of eigenvalues